Particle systems

position $\vec{x}$

velocity $\vec{v} = \dot{\vec{x}}$

acceleration $\dot{\vec{v}}$

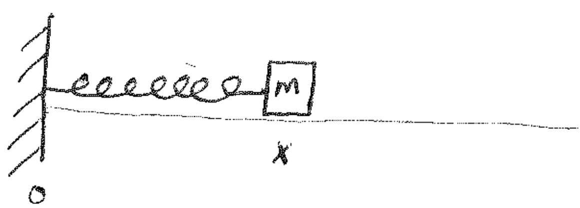
Newton's law: $m\dot{\vec{v}} = \vec{f}$

mass: m

Simple force

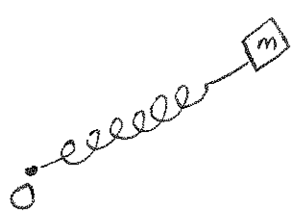
gravity $\vec{f} = m\vec{g} = m\begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix}$ $\qquad g \approx 9.8\, m\, s^{-2}$

spring (1D)



$$m\dot{v} = f = -k\left(\frac{x}{\ell} - 1\right)$$

Spring (3D)



$$m\dot{\vec{v}} = \vec{f} = -k\left(\frac{\|x\|}{\ell} - 1\right)\frac{x}{\|x\|}$$

Simulation

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{m} f(x,v)$$

↑ general force

Forward Euler

$t = 0, \Delta t, 2\Delta t, 3\Delta t, \ldots$

$$\dot{x} \approx \frac{x^{n+1} - x^n}{\Delta t} \qquad \frac{x^{n+1} - x^n}{\Delta t} = v^n \Rightarrow x^{n+1} = x^n + \Delta t\, v^n$$

$$\dot{v} \approx \frac{v^{n+1} - v^n}{\Delta t} \qquad \frac{v^{n+1} - v^n}{\Delta t} = \frac{1}{m} f(x^n, v^n) \Rightarrow v^{n+1} = v^n + \frac{\Delta t}{m} f(x^n, v^n)$$

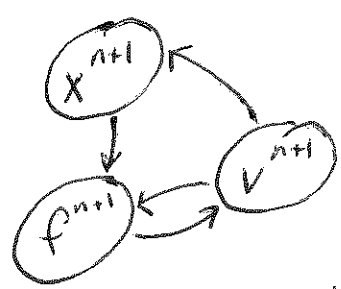
* easy

Backward Euler

$$\frac{x^{n+1} - x^n}{\Delta t} = v^{n+1}$$

$$\frac{v^{n+1} - v^n}{\Delta t} = \frac{1}{m} f(x^{n+1}, v^{n+1})$$

* stable



nonlinear system of equations

Midpoint rule

$$\frac{x^{n+1} - x^n}{\Delta t} = v^{n+\frac{1}{2}}$$

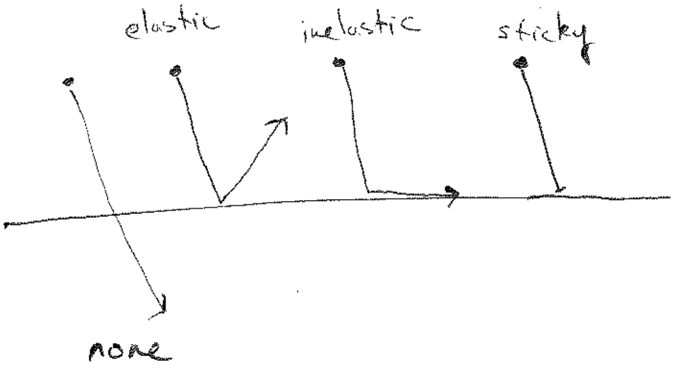
$$\frac{v^{n+1} - v^n}{\Delta t} = \frac{1}{m} f(x^{n+\frac{1}{2}}, v^{n+\frac{1}{2}})$$

$$x^{n+\frac{1}{2}} = \frac{x^{n+1} + x^n}{2} \qquad v^{n+\frac{1}{2}} = \frac{v^{n+1} + v^n}{2}$$

* accurate

Collision

x^{n+1} is inside object
* eg. below ground

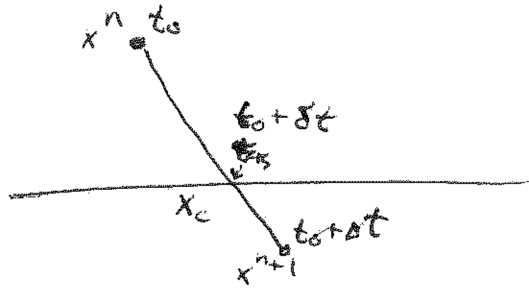


* what do you do?

* collision at time $\delta t + t_0$

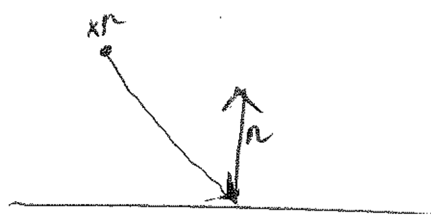
* $X_c = X^n + \delta t\, v$

v^n or v^{n+1} or $v^{n+\frac{1}{2}}$



* what about velocity?

* collision normal $= \vec{n}$ (outward)



* velocity points in → fix it

* $\vec{V} = \vec{V_t} + \vec{n}\, V_n$

tangential normal

sticky → $\vec{V}' \leftarrow \vec{0}$

inelastic → $V_n' \leftarrow 0$ $\qquad \vec{V}' \leftarrow \vec{V_t}$

elastic → $V_n' \leftarrow -V_n$ $\qquad \vec{V}' \leftarrow \vec{V_t} - \vec{n}\, V_n$

* friction?

* we have made many errors

  → did not move object for $\Delta t - \delta t$

    → can move rest of time after fixing v

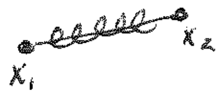
  → $\dfrac{\vec{x}^{n+1} - x^{n}}{\Delta t} \approx \dot{\vec{x}}$

* these vanish if $\Delta t \to 0$

* if more accuracy required, use smaller δt, slower

multiple particles

particle 1	particle 2
x_1^n	x_2^n
v_1^n	v_2^n



$$\frac{x_1^{n+1} - x_1^n}{\Delta t} = v_1^n \qquad \frac{x_2^{n+1} - x_2^n}{\Delta t} = v_2^n$$

$$\frac{v_1^{n+1} - v_1^n}{\Delta t} = \frac{1}{m_1} f_1(x_1^n, x_2^n, v_1^n, v_2^n) \leftarrow$$ particles affect each other

$$\frac{v_2^{n+1} - v_2^n}{\Delta t} = \frac{1}{m_2} f_2(x_1^n, x_2^n, v_1^n, v_2^n)$$

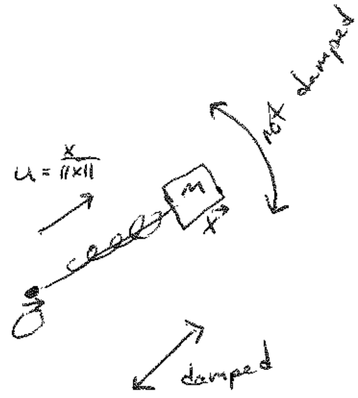
Newton's third law: $f_1 = -f_2$

e.g. $f_1 = -f_2 = -k\left(\frac{\|x_1 - x_2\|}{\ell} - 1\right)\frac{x_1 - x_2}{\|x_1 - x_2\|}$

Damping

* energy loss to heat

* force opposing <u>motion</u>



* force proportional to $\vec{v}\cdot\vec{u}$

* direction of $\vec{u}$

* $f = \pm (u\cdot v)u$

* sign: $\quad m\dot{v} = \pm(u\cdot v)u$

$+v \Rightarrow -\dot{v} \quad \Rightarrow \ -$ sign

* $f = -(v\cdot u)u$

$$f = -k\left(\frac{\|x\|}{\ell} - 1\right)\frac{x}{\|x\|} - c\left(v\cdot\frac{x}{\|x\|}\right)\frac{x}{\|x\|}$$

* $f(x, v)$