CS141: Intermediate Data Structures and Algorithms

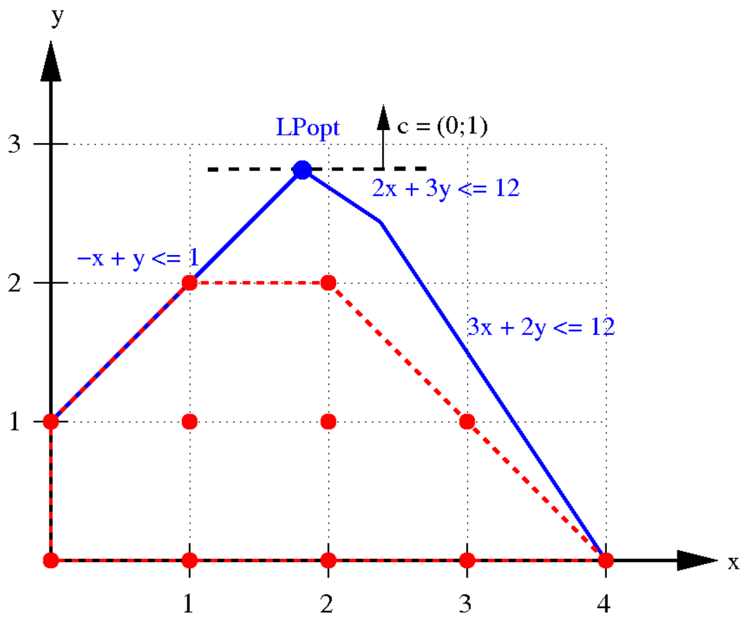
Dynamic Programming

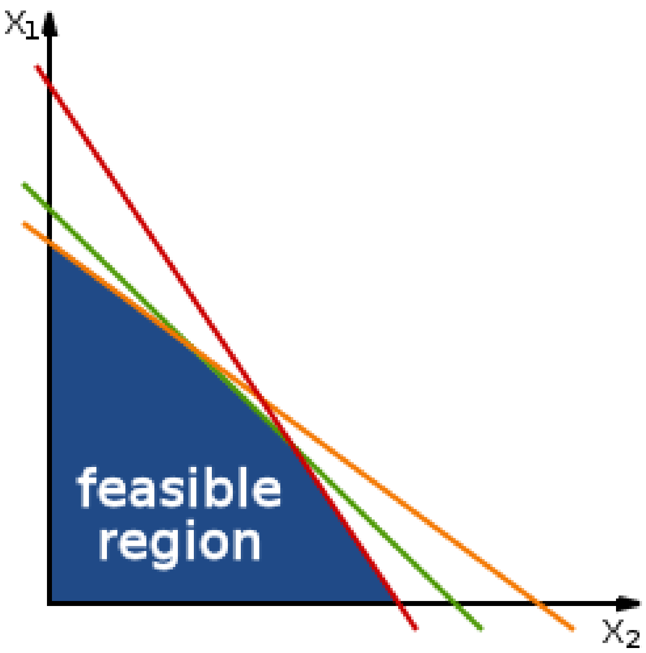
Amr Magdy

Programming?

- In this context, programming is a tabular method

- Other examples:
  - Linear programing
  - Integer programming





Rod Cutting Problem

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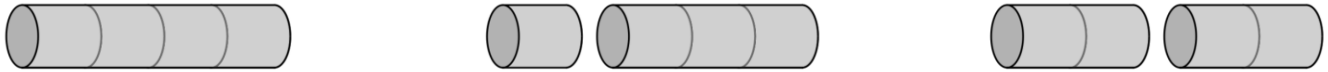

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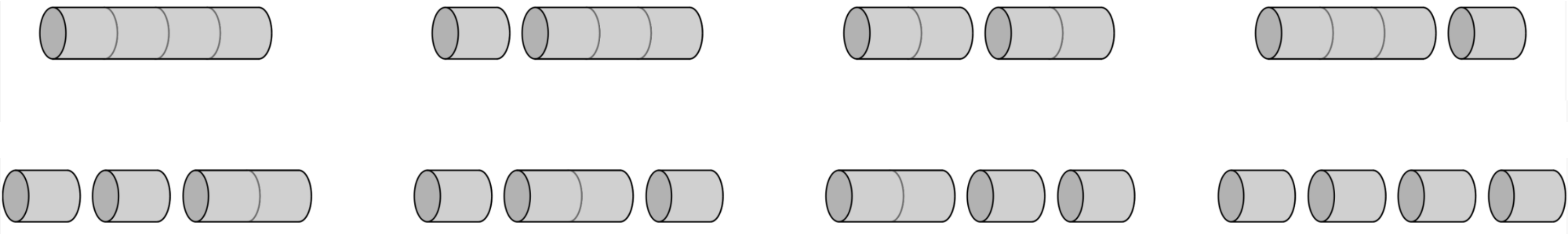

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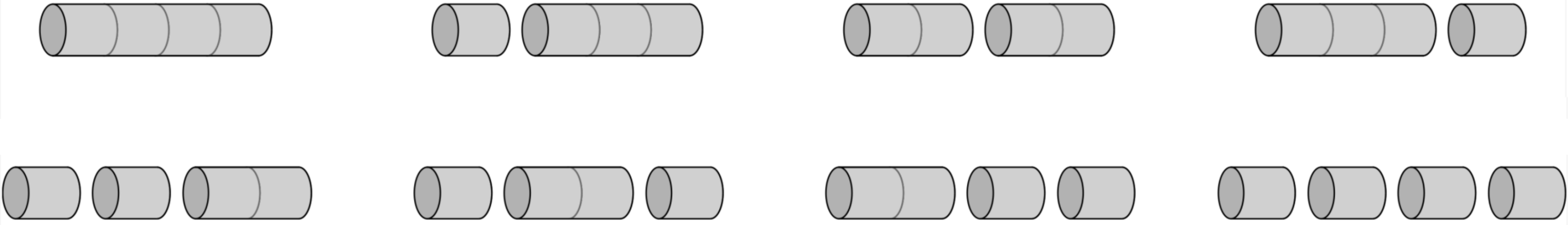

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

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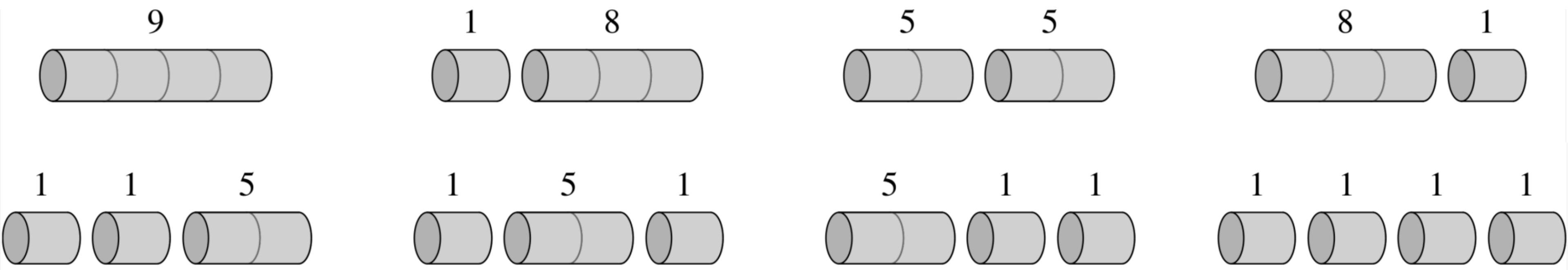

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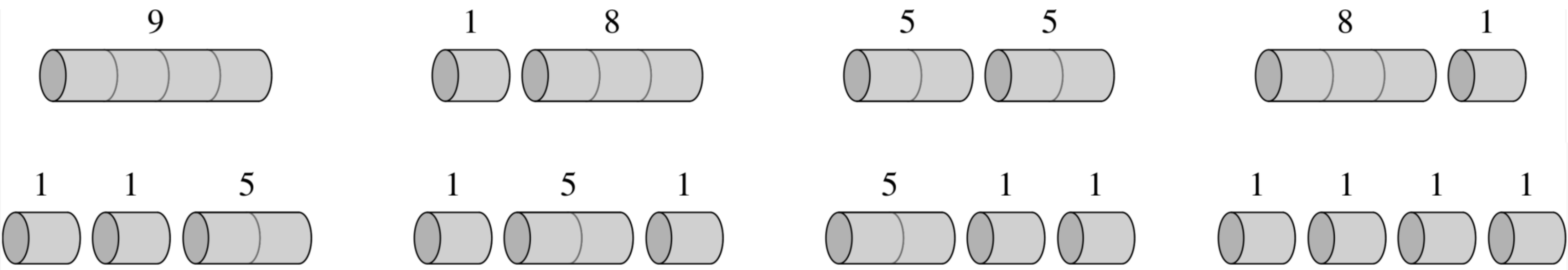

length i	1	2	3	4	5	6	7	8	9	10
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- Given a rod of length n and prices p_i, find the cutting strategy that makes the maximum revenue
  - In the example: (2+2) cutting makes r=5+5=10

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    - Total: $O(n^n)$

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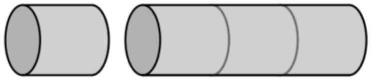
divide



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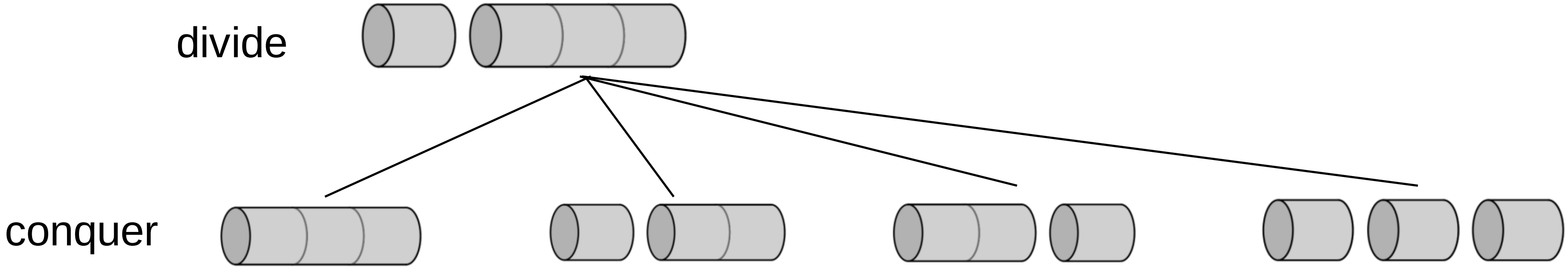

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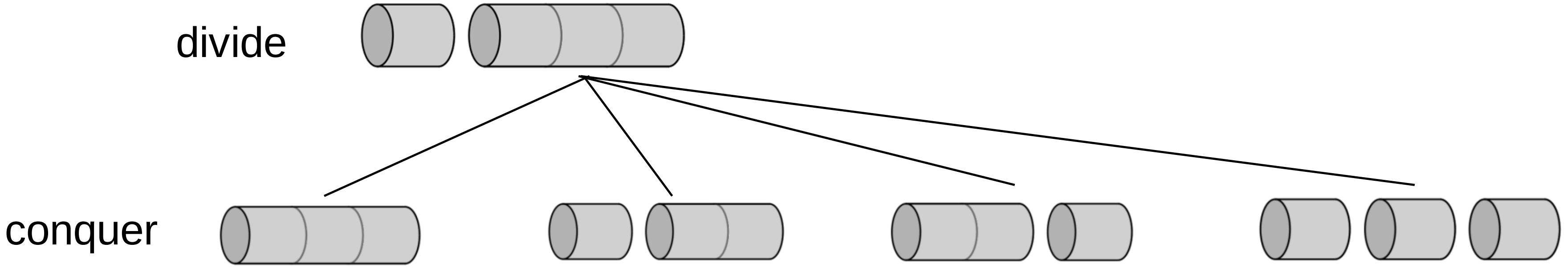

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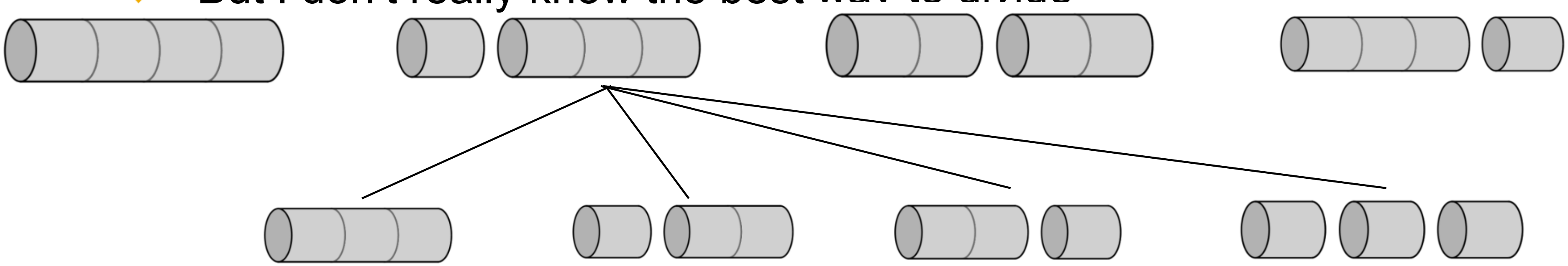

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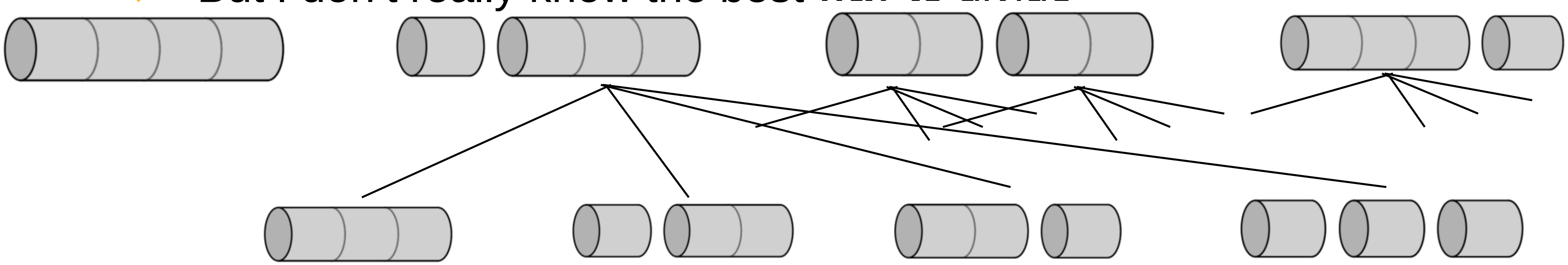

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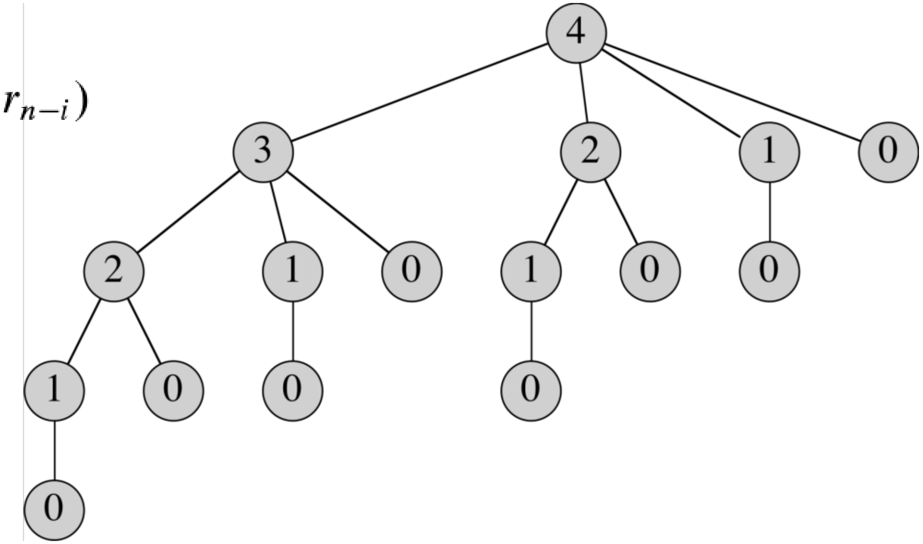
$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$



Rod Cutting Problem

- Recursive top-down algorithm

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

```
CUT-ROD(p, n)
if n == 0
    return 0
q = -∞
for i = 1 to n
    q = max(q, p[i] + CUT-ROD(p, n - i))
return q
```

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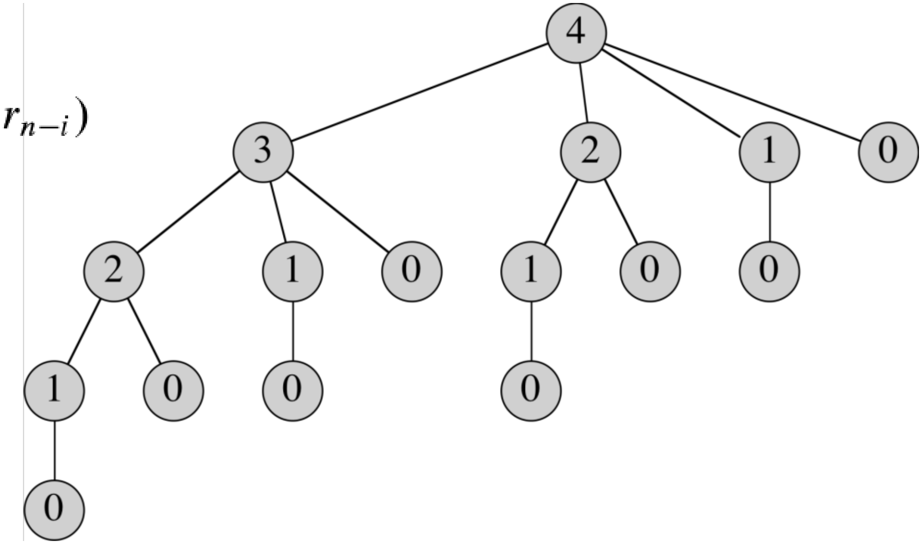

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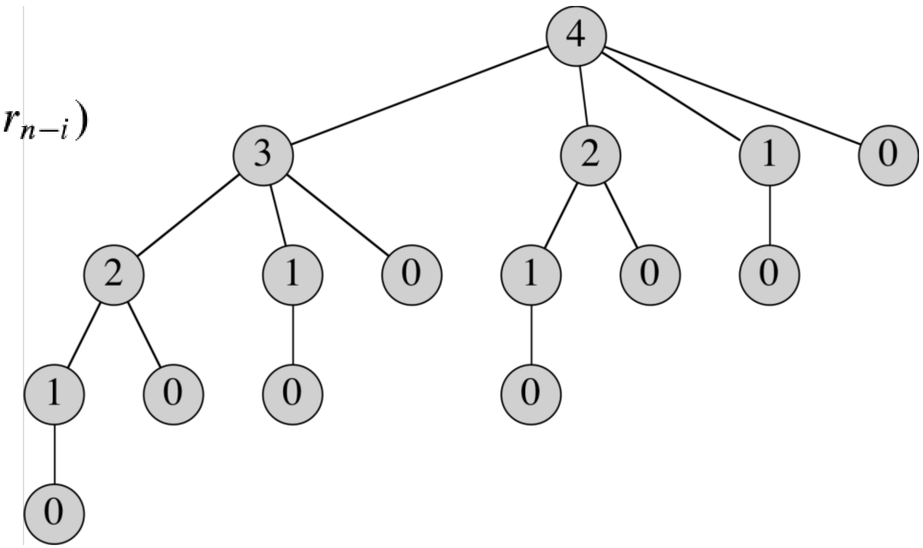

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$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) .$$

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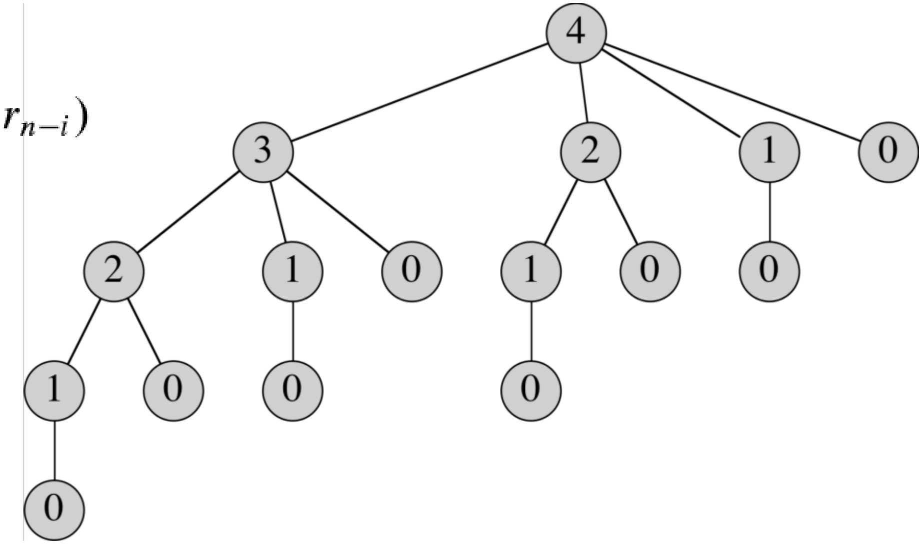

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$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) \,.$$

$$T(n) = 2^n$$

(Prove by induction)

Rod Cutting Recursive Complexity

- Find the complexity of $\mathrm{T(n)} = 1 + \sum_{j=0}^{n-1} T(j)$
- Proof by induction:
  - Assume the solution is some function X(n)
  - Show that X(n) is true for the smallest n (the base case), e.g., n=0
  - Prove that X(n+1) is a solution for T(n+1) given X(n)
  - You are done
- Given $\mathrm{T(n)} = 1 + \sum_{j=0}^{n-1} T(j)$
- Assume $\mathrm{T(n)} = 2^n$
- $\mathrm{T(0)} = 1 + \sum_{j=0}^{-1} T(j) = 1 = 2^0$ (base case)
- $\mathrm{T(n+1)} = 1 + \sum_{j=0}^{n} T(j) = 1 + \sum_{j=0}^{n-1} T(j) + T(n) = T(n) + T(n) = 2T(n) = 2 * 2^n = 2^{n+1}$
- Then, $\mathrm{T(n)} = 2^n$

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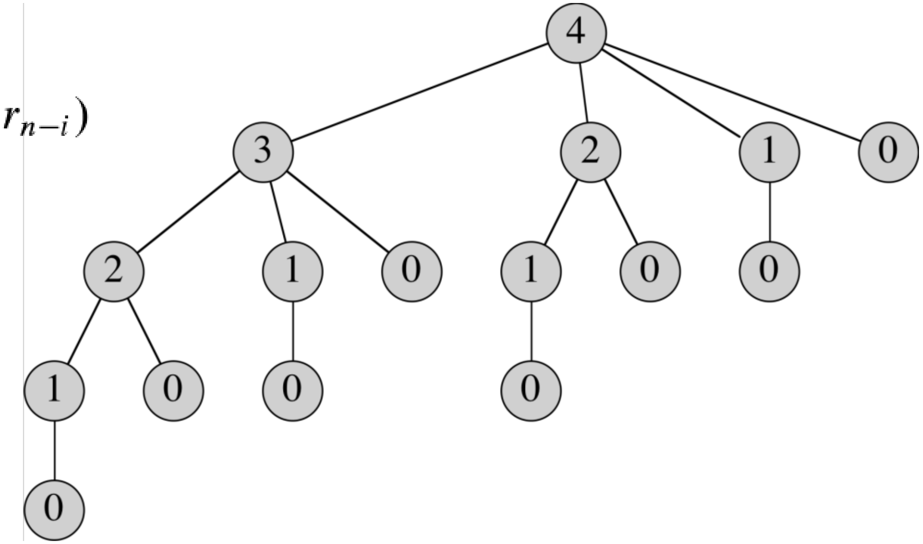

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$$T(n) = 2^n$$ (Prove by induction)

- Can we do better?

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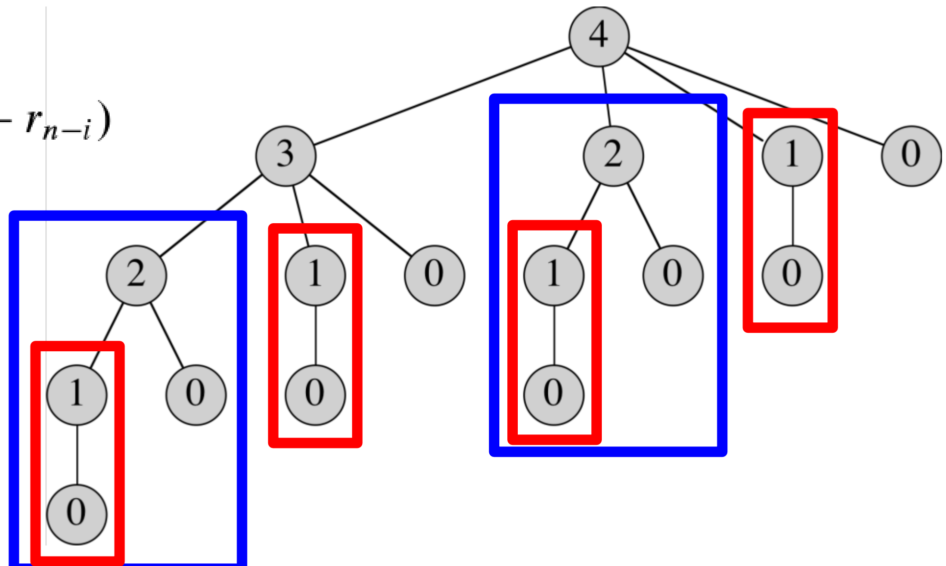

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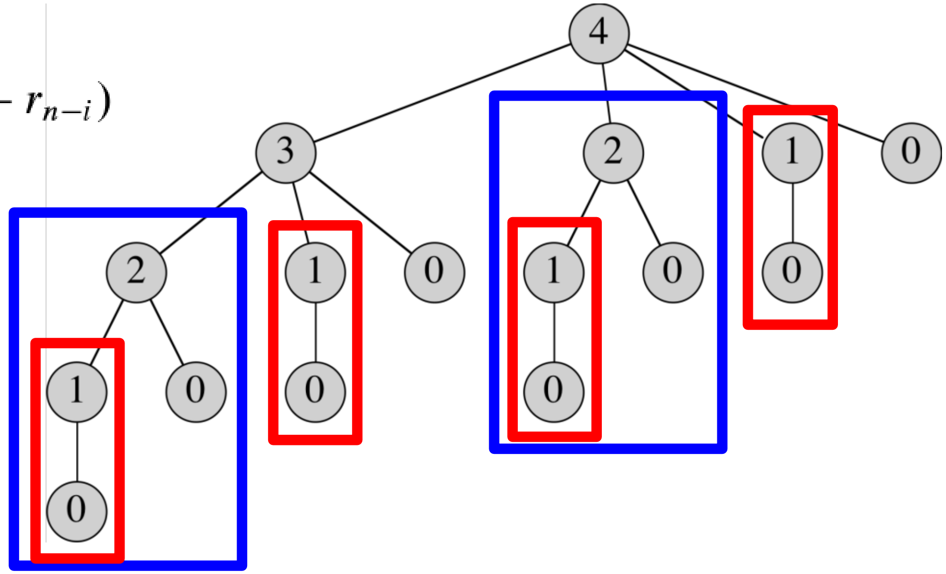

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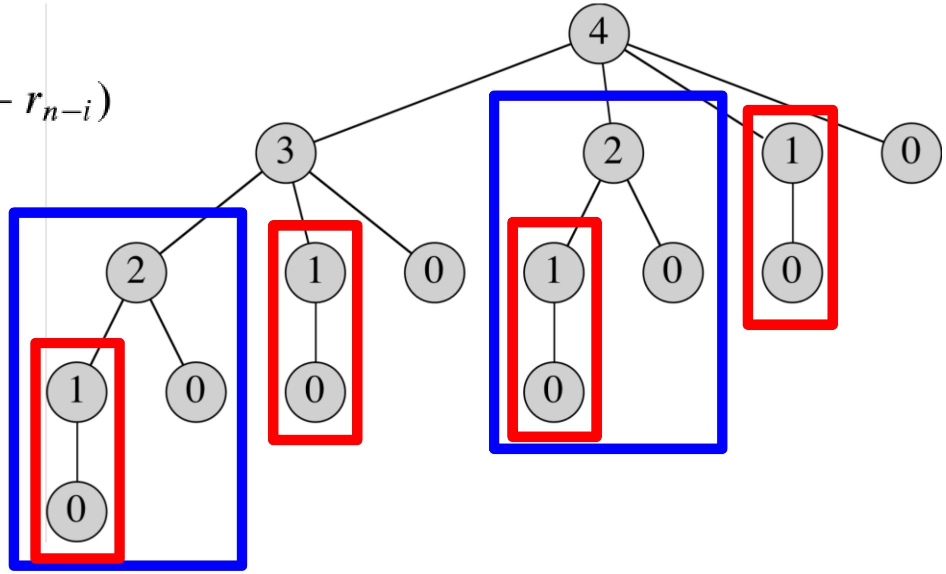

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  - Solve each subproblem once
  - Write down the solution in a lookup table (array, hashtable,…etc)
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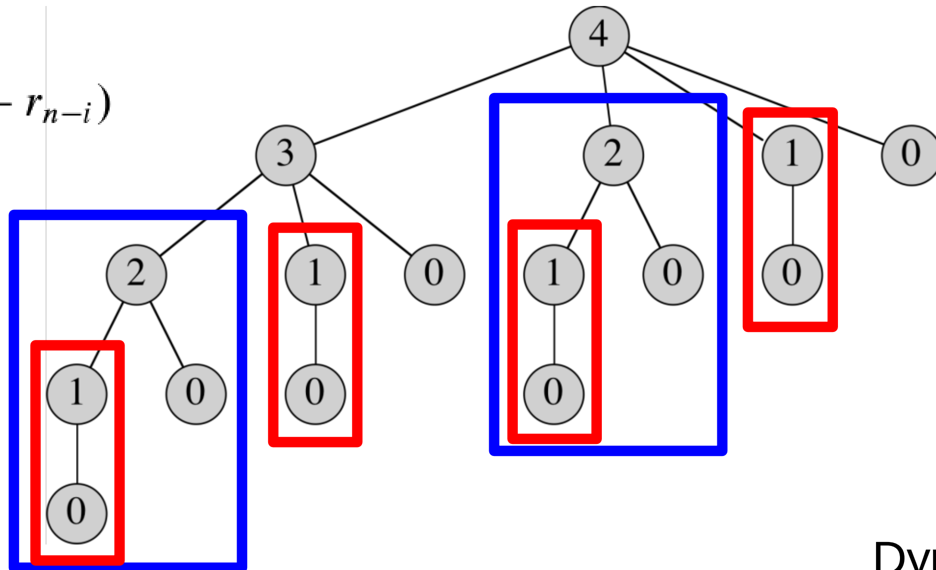

Dynamic Programming

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Rod Cutting Problem

- Recursive top-down dynamic programming algorithm

```
MEMOIZED-CUT-ROD(p, n)
let r[0..n] be a new array
for i = 0 to n
    r[i] = −∞
return MEMOIZED-CUT-ROD-AUX(p, n, r)
```

```
MEMOIZED-CUT-ROD-AUX(p, n, r)
if r[n] ≥ 0
    return r[n]
if n == 0
    q = 0
else q = −∞
    for i = 1 to n
        q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
r[n] = q
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r[n] = q
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$\Theta(n^2)$

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BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        q = max(q, p[i] + r[j - i])
    r[j] = q
return r[n]
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Elements of a Dynamic Programming Problem

- Optimal substructure
  - Optimal solution of a larger problem comes from optimal solutions of smaller problems
- Subproblem overlapping
  - Same exact sub-problems are solved again and again

Dynamic Programming vs. D&C

- How different?

Dynamic Programming vs. D&C

- How different?
  - No subproblem overlapping
    - Each subproblem with distinct input is a new problem
  - Not necessarily optimization problems, i.e., no objective function

Reconstructing Solution

- Rod cutting problem: What are the actual cuts?
  - Not only the best revenue (the optimal objective function value)

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```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] and s[1..n] be new arrays
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        if q < p[i] + r[j - i]
            q = p[i] + r[j - i]
            s[j] = i
    r[j] = q
return r and s
```

Reconstructing Solution

- Rod cutting problem: What are the actual cuts?
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```
PRINT-CUT-ROD-SOLUTION(p, n)
(r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while n > 0
    print s[n]
    n = n - s[n]
```

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$		1	2	3	2	2	6	1	2	3	10

  - Let’s trace examples

Matrix Chain Multiplication

- How to multiply a chain of four matrices $A_1 A_2 A_3 A_4$?

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$$((A_1A_2)(A_3A_4))$$
$$((A_1(A_2A_3))A_4)$$
$$(((A_1A_2)A_3)A_4)$$

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```
MATRIX-MULTIPLY(A, B)
if A.columns ≠ B.rows
    error "incompatible dimensions"
else let C be a new A.rows × B.columns matrix
    for i = 1 to A.rows
        for j = 1 to B.columns
            c_ij = 0
            for k = 1 to A.columns
                c_ij = c_ij + a_ik · b_kj
    return C
```

- Does it really make a difference?
- # of multiplications:

A.rows*B.cols*A.cols

Matrix Chain Multiplication

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- # of multiplications:

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- Example:

A1*A2*A3

Dimensions:

10x100x5x50

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- # of multiplications in ((A1*A2)*A3)=10*100*5+10*5*50=7.5K
- # of multiplications in (A1*(A2*A3))=100*5*50+10*100*50=75K

Matrix Chain Multiplication

- Given n matrices A_1 A_2 … A_n of dimensions p_0 p_1 … p_n, find the optimal parentheses to multiply the matrix chain

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- $(A_1 A_2 A_3)(A_4 A_5 \ldots A_n)$

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- Sub-chains C1 = $(A_1 A_2 A_3)$, C2 = $(A_4 A_5 \dots A_n)$

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- A_1 A_2 A_3 A_4 A_5 … A_n
- $(A_1$ A_2 $A_3)(A_4$ A_5 … $A_n)$
- Sub-chains C1 = $(A_1$ A_2 $A_3)$, C2 = $(A_4$ A_5 … $A_n)$
- Total Cost C = cost(C1)+cost(C2)+$p_0p_3p_n$

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- Total Cost C = cost(C1)+cost(C2)+$p_0 p_3 p_n$
- Then, if cost(C1) and cost(C2) are minimal (i.e., optimal), then C is optimal (optimal substructure holds)

Matrix Chain Multiplication

- Given n matrices $A_1\ A_2\ \ldots\ A_n$ of dimensions $p_0\ p_1\ \ldots\ p_n$, find the optimal parentheses to multiply the matrix chain
- $A_1\ A_2\ A_3\ A_4\ A_5\ \ldots\ A_n$
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- Total Cost $C = cost(C1)+cost(C2)+p_0p_3p_n$
- Then, if cost(C1) and cost(C2) are minimal (i.e., optimal), then C is optimal (optimal substructure holds)
- Proof by contradiction:
  - Given C is optimal, are cost(C1)=c1 and cost(C2)=c2 optimal?
  - Assume c1 is NOT optimal, then ∃ an optimal solution of cost c1' < c1
  - Then c1'+c2+p < c1+c2+p → C' < C
  - Then C is not optimal → contradiction!
  - Then C1 has to be optimal → optimal substructure holds

Matrix Chain Multiplication

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- Optimal C1, C2 might be one of different options
  - $C1 = (A_1\ A_2)$, $C2 = (A_3\ A_4\ A_5\ \ldots\ A_n)$
  - $C1 = (A_1)$, $C2 = (A_2\ A_3\ A_4\ A_5\ \ldots\ A_n)$
  - $C1 = (A_1\ A_2\ A_3\ A_4)$, $C2 = (A_5\ \ldots\ A_n)$
  - …….

Matrix Chain Multiplication

- Generally: $A_i \dots A_k \dots A_j$ of dimensions $p_i \dots p_k \dots p_j$
- $(A_i \dots A_k)(A_{k+1} \dots A_j)$, where $k=i,i+1,\dots j-1$
- Then, solve each sub-chains recursively

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$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

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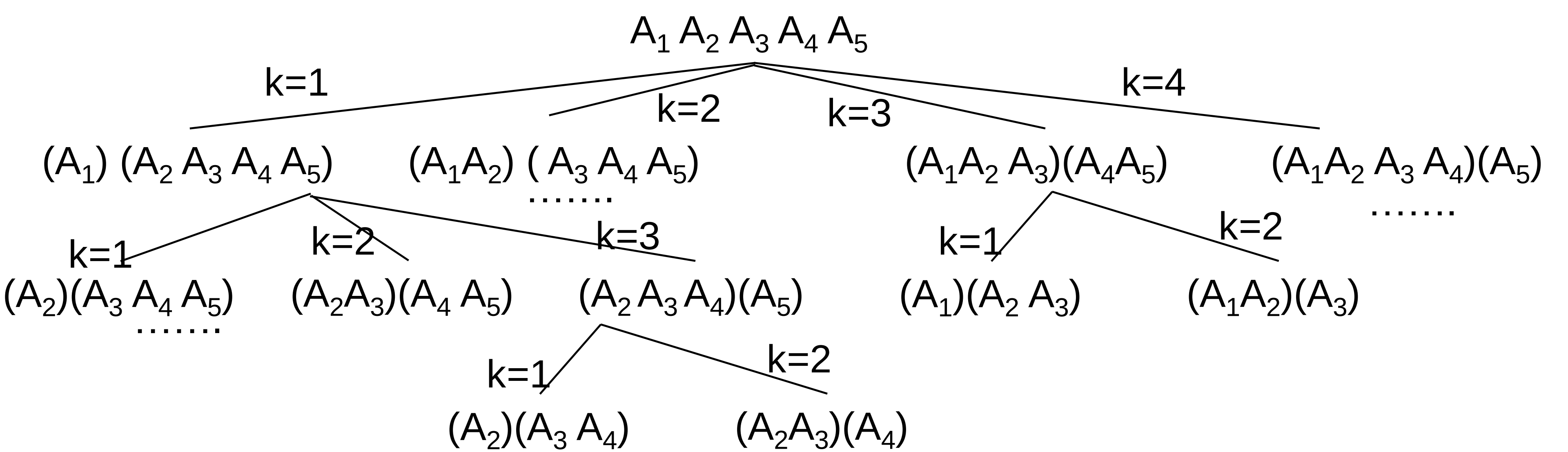

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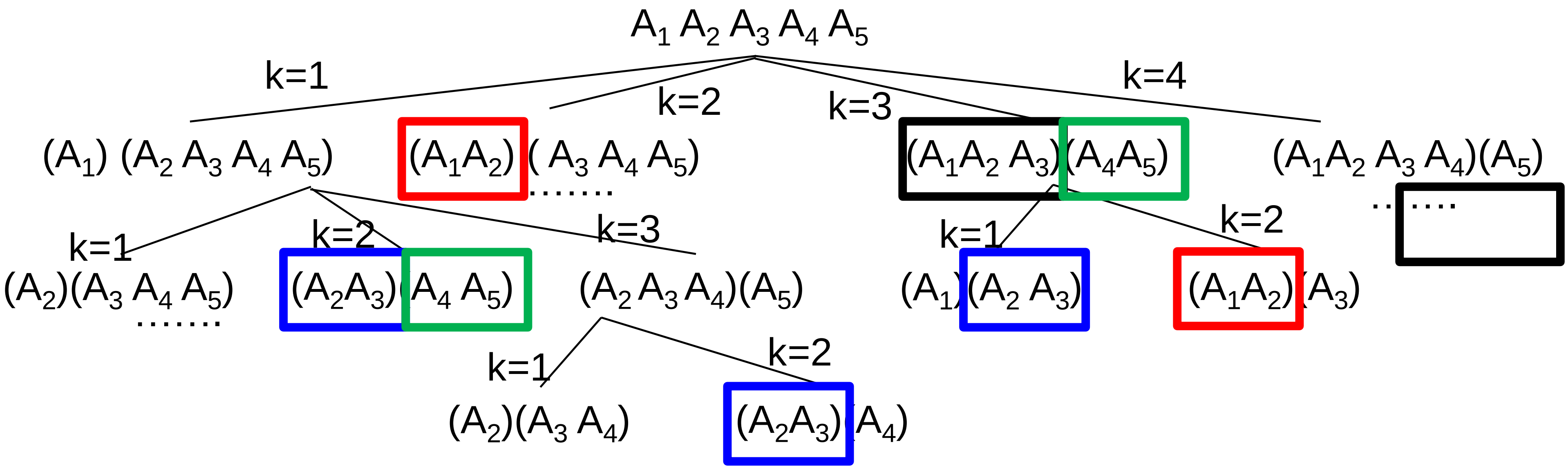

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- Obviously, a lot of overlapping subproblems appear

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- Obviously, a lot of overlapping subproblems appear
- Optimal substructure + subproblem overlapping = dynamic programming

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?

Matrix Chain Multiplication: Designing Algorithm

- What is the smallest subproblem?
  - A chain of length 2

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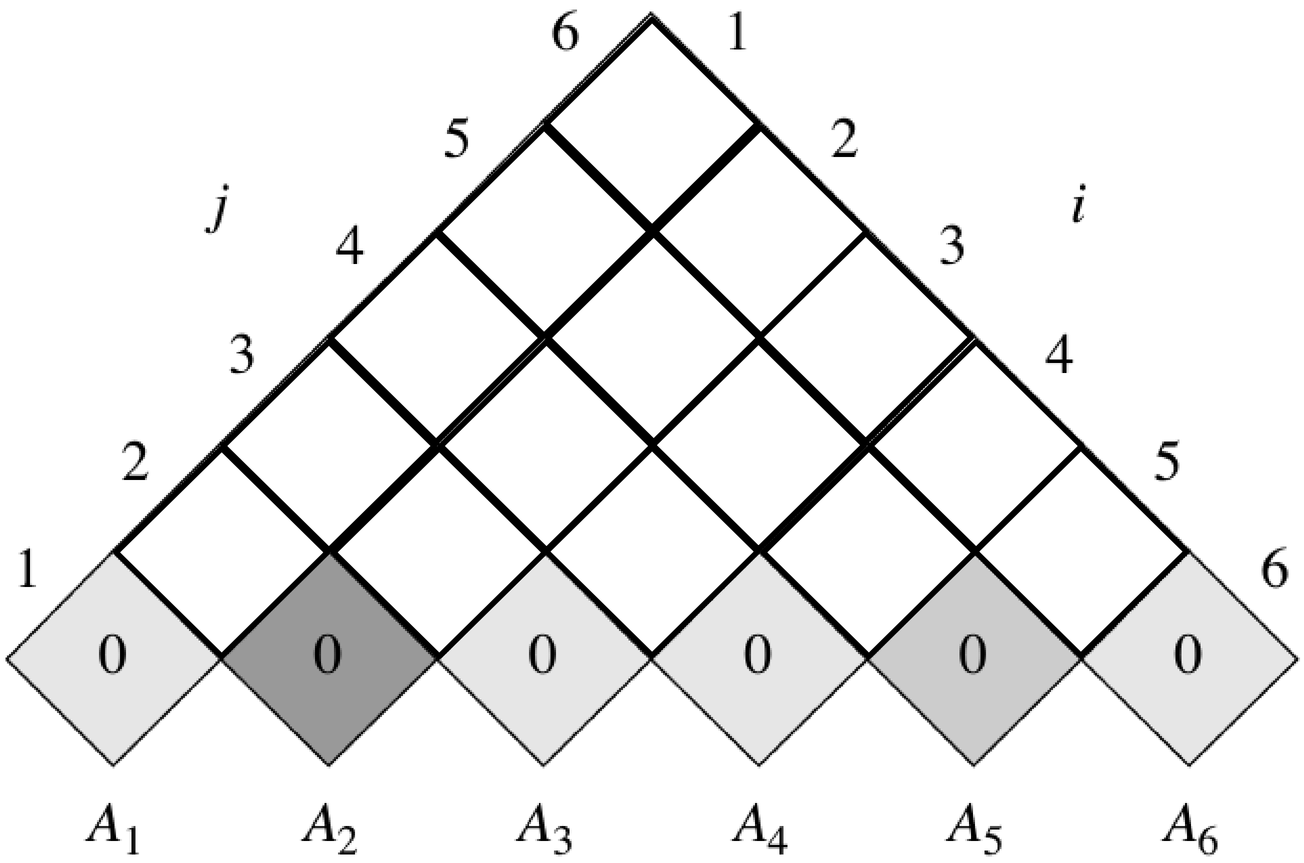

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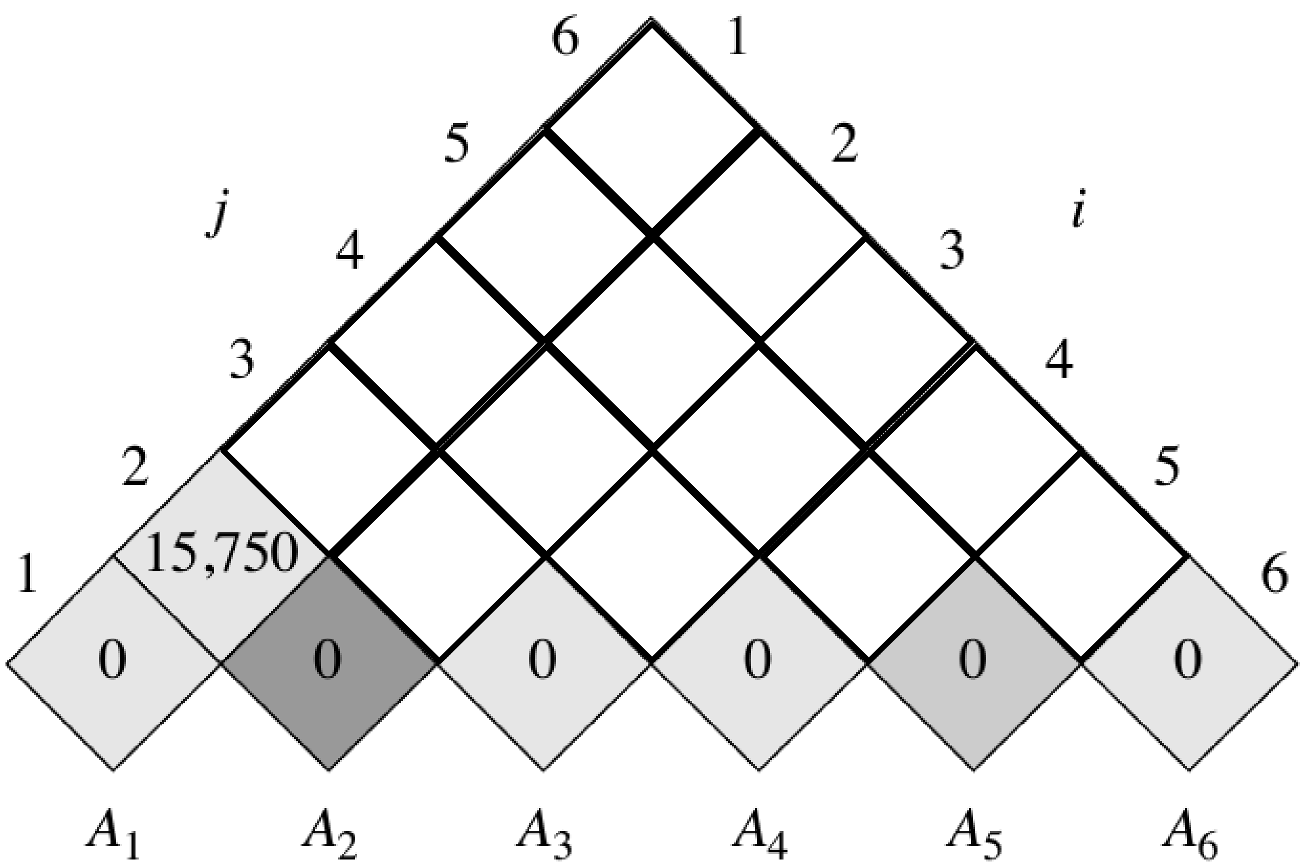

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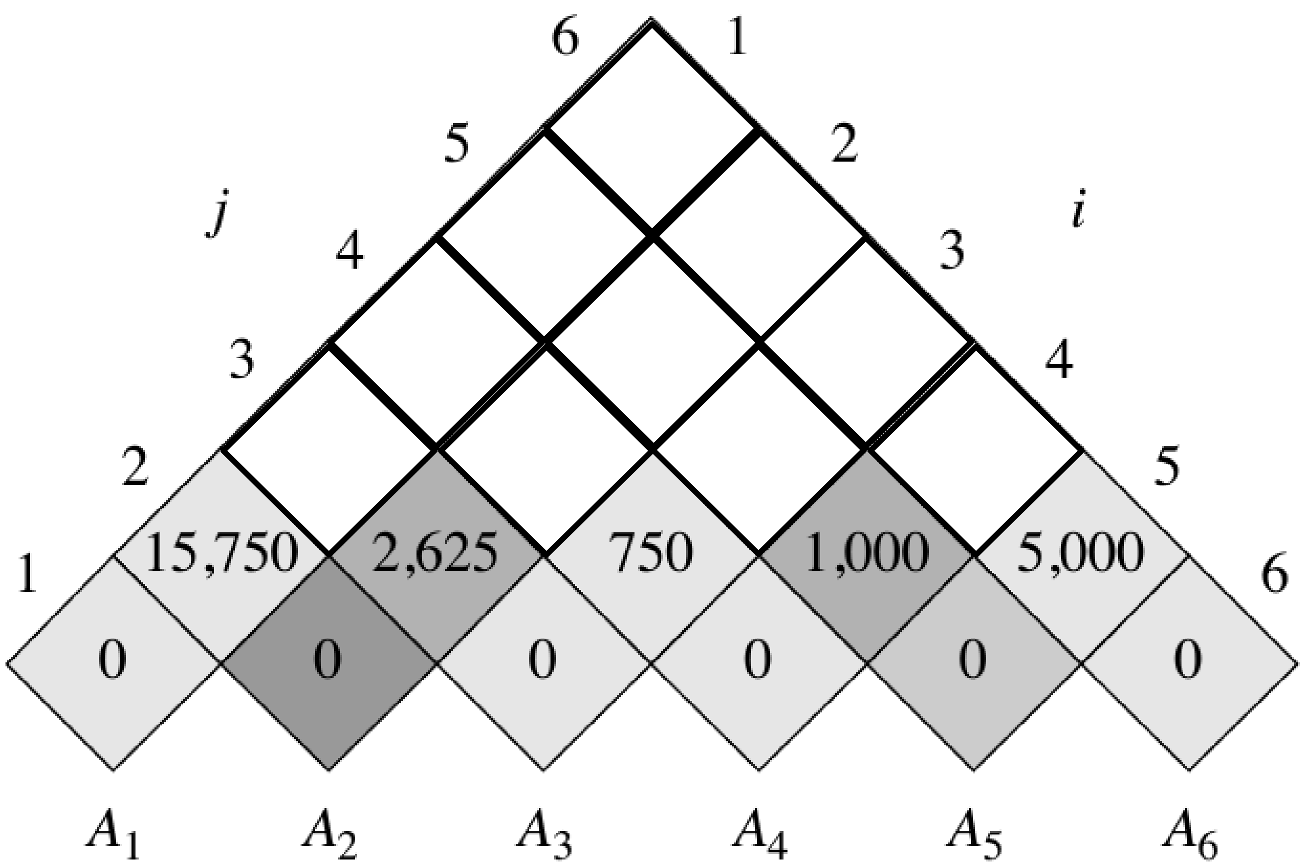

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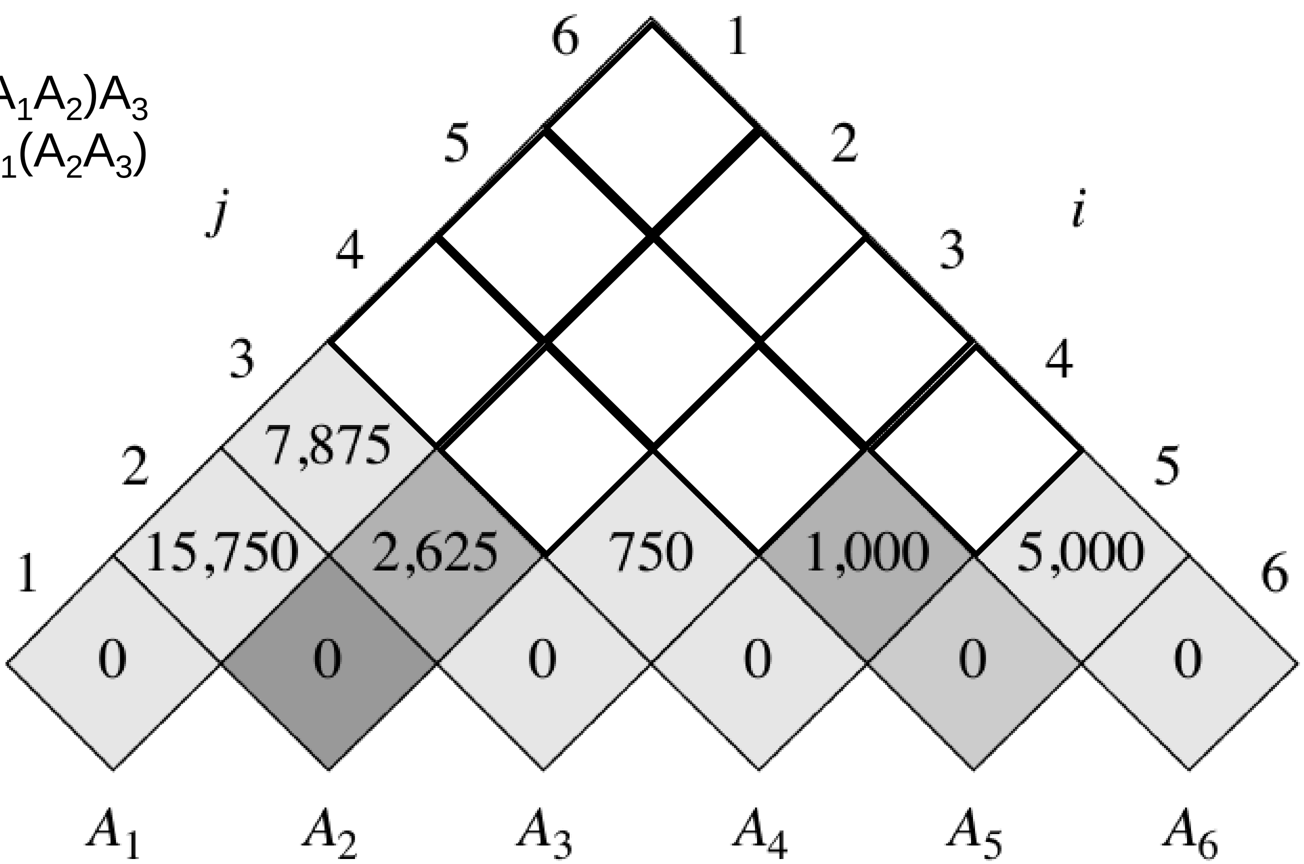
$A_1A_2A_3 = (A_1A_2)A_3$
Or $A_1(A_2A_3)$



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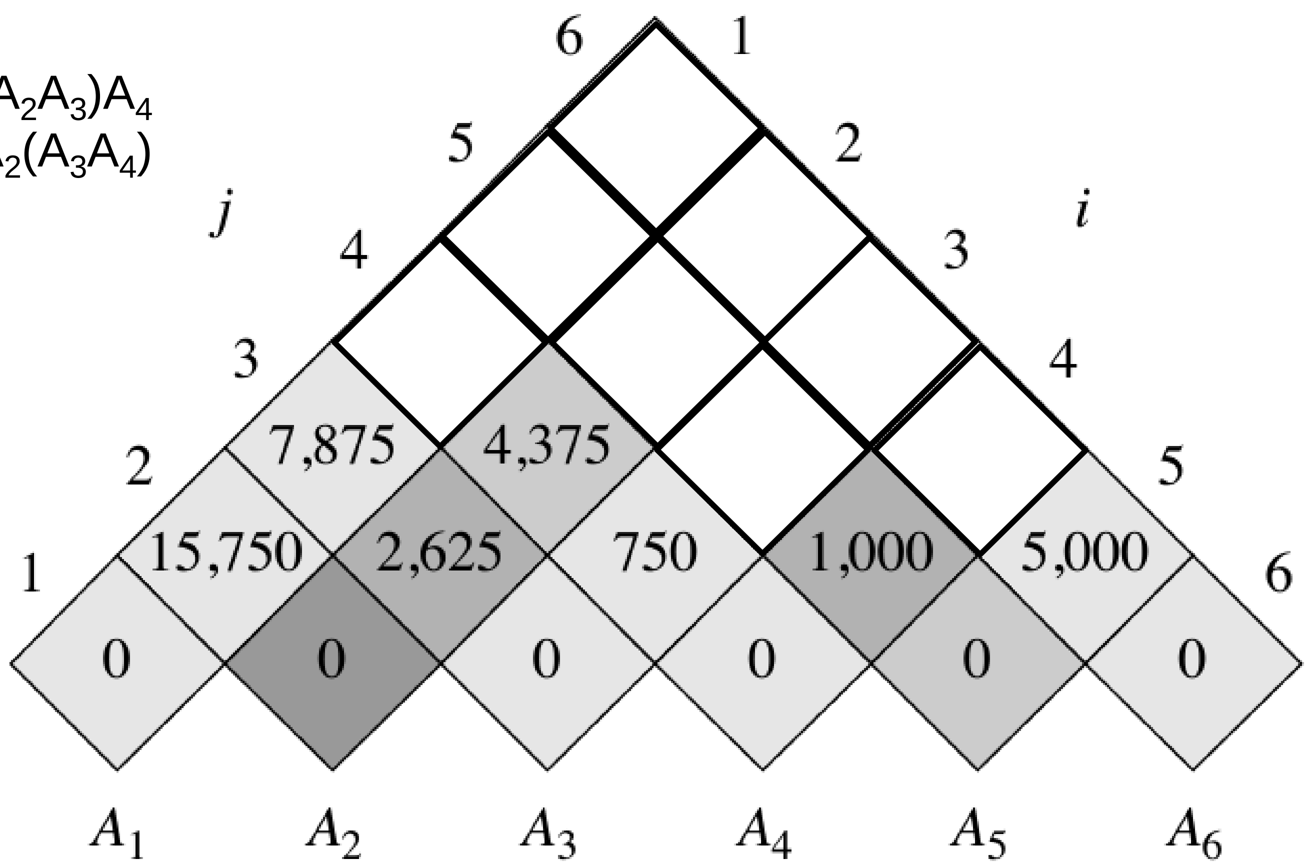
$A_2A_3A_4 = (A_2A_3)A_4$
Or $A_2(A_3A_4)$



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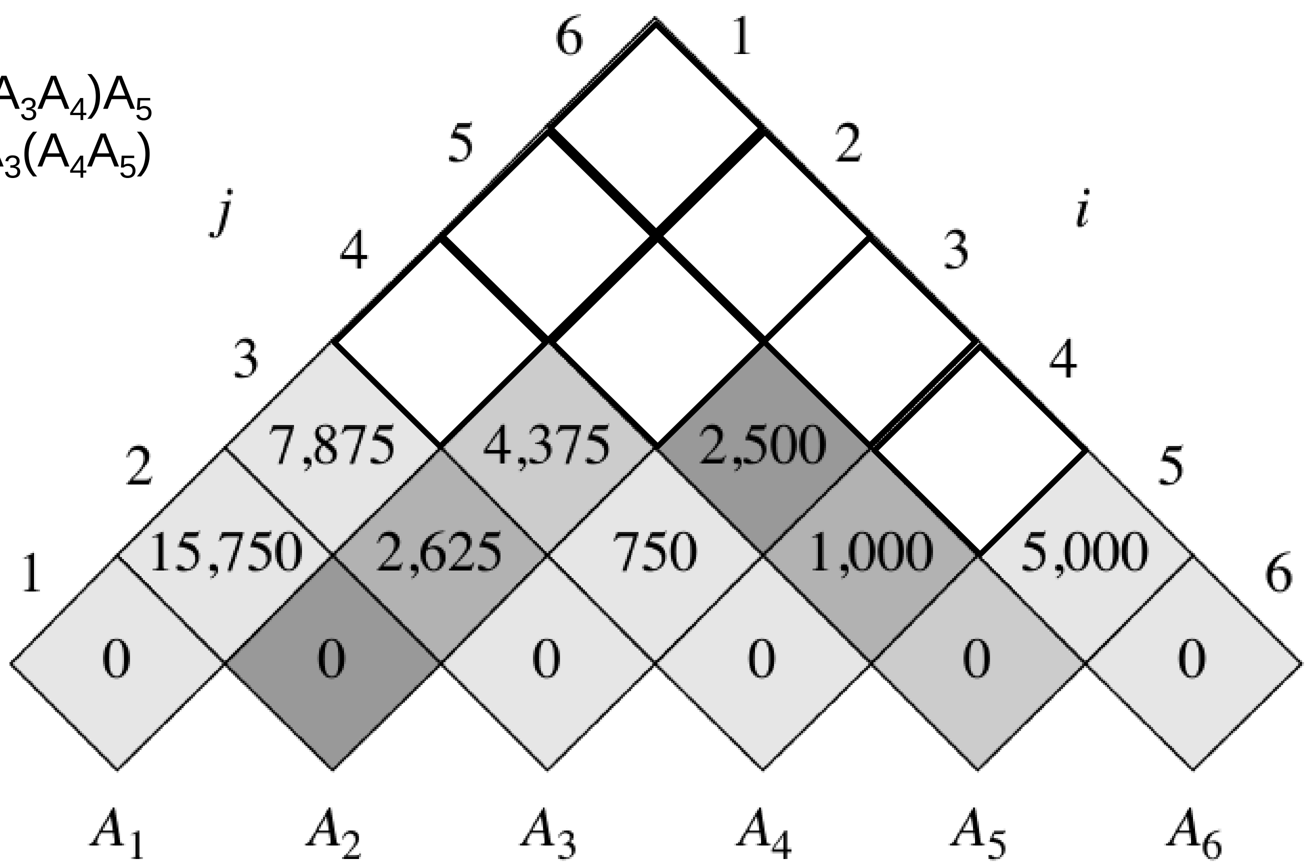
$A_3A_4A_5 = (A_3A_4)A_5$
Or $A_3(A_4A_5)$



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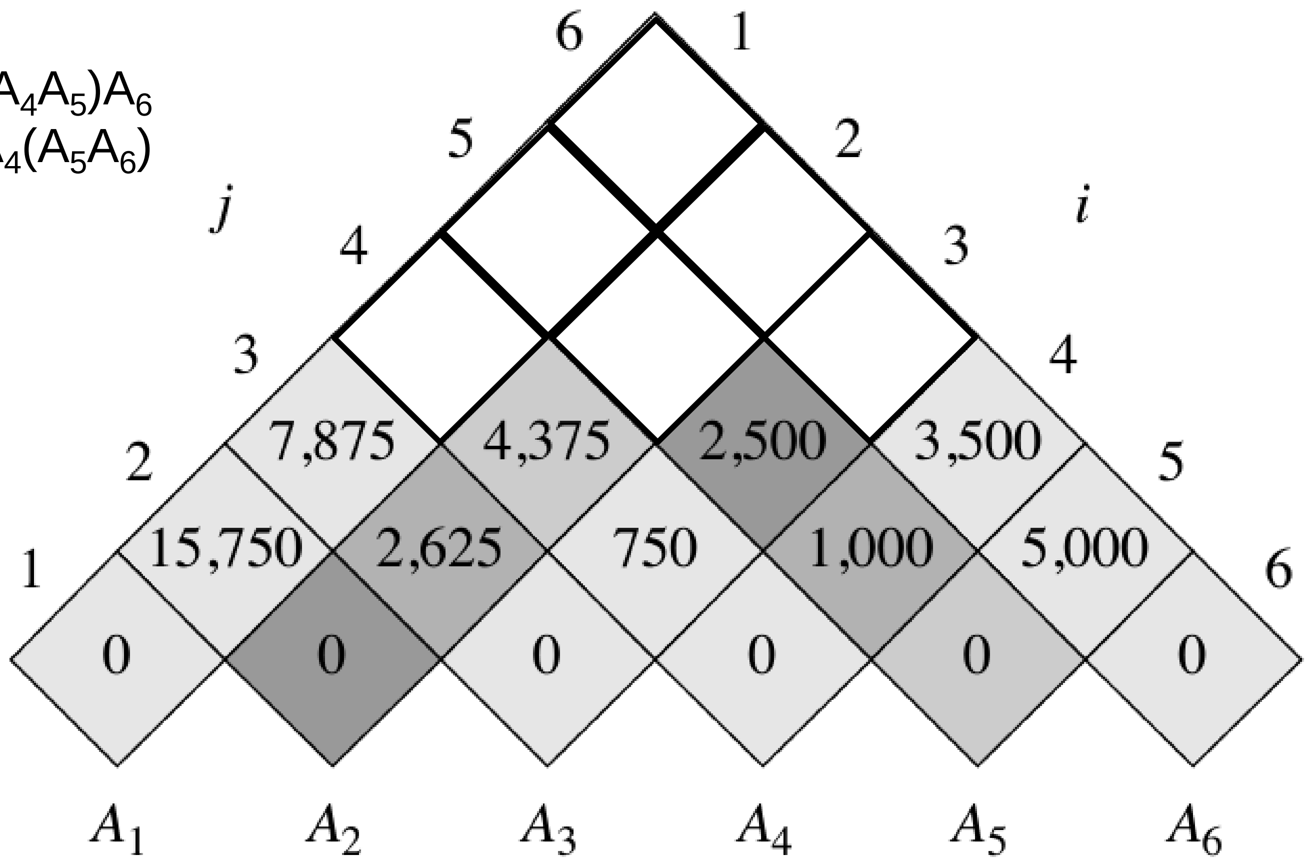
$A_4A_5A_6 = (A_4A_5)A_6$

Or $A_4(A_5A_6)$



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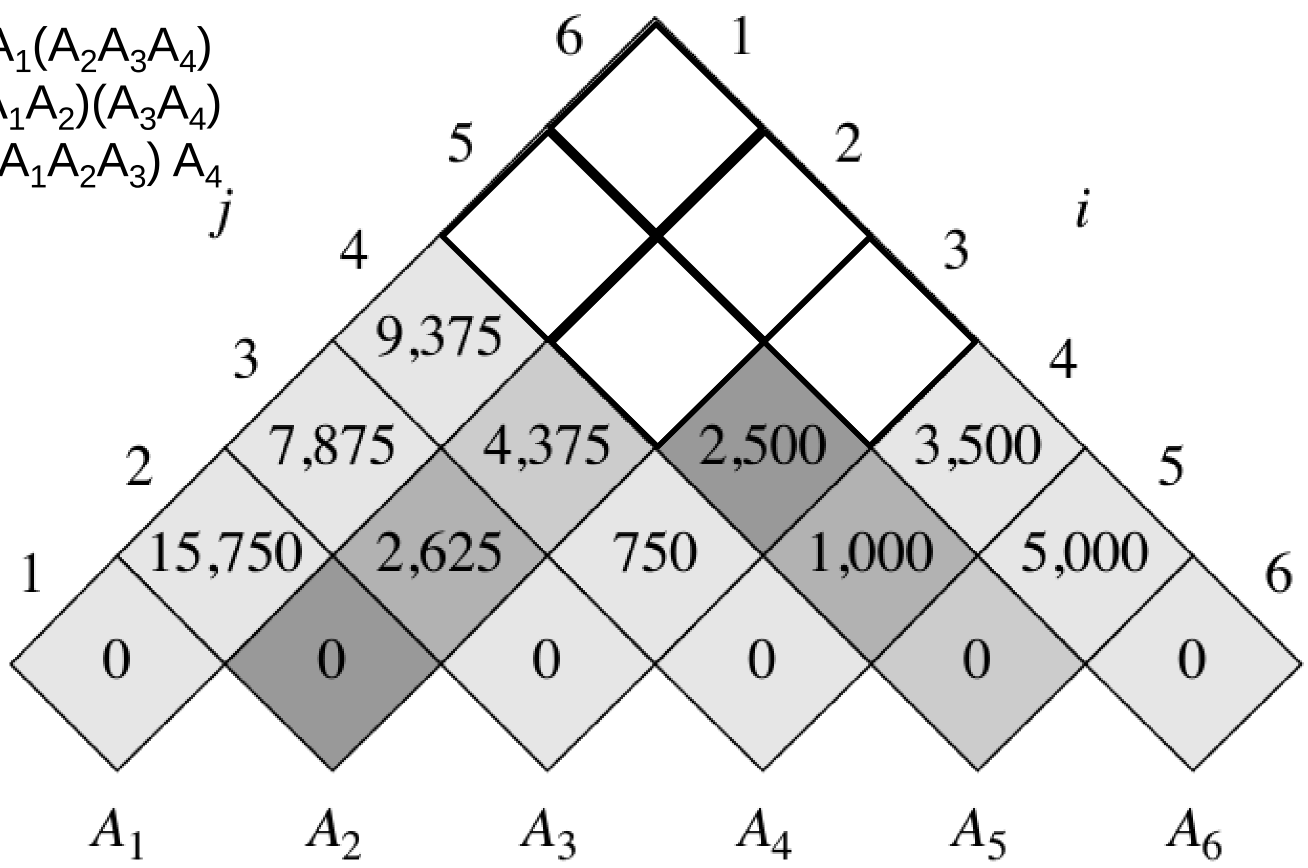
$A_1A_2A_3A_4 = A_1(A_2A_3A_4)$
Or $(A_1A_2)(A_3A_4)$
Or $(A_1A_2A_3)\ A_4$



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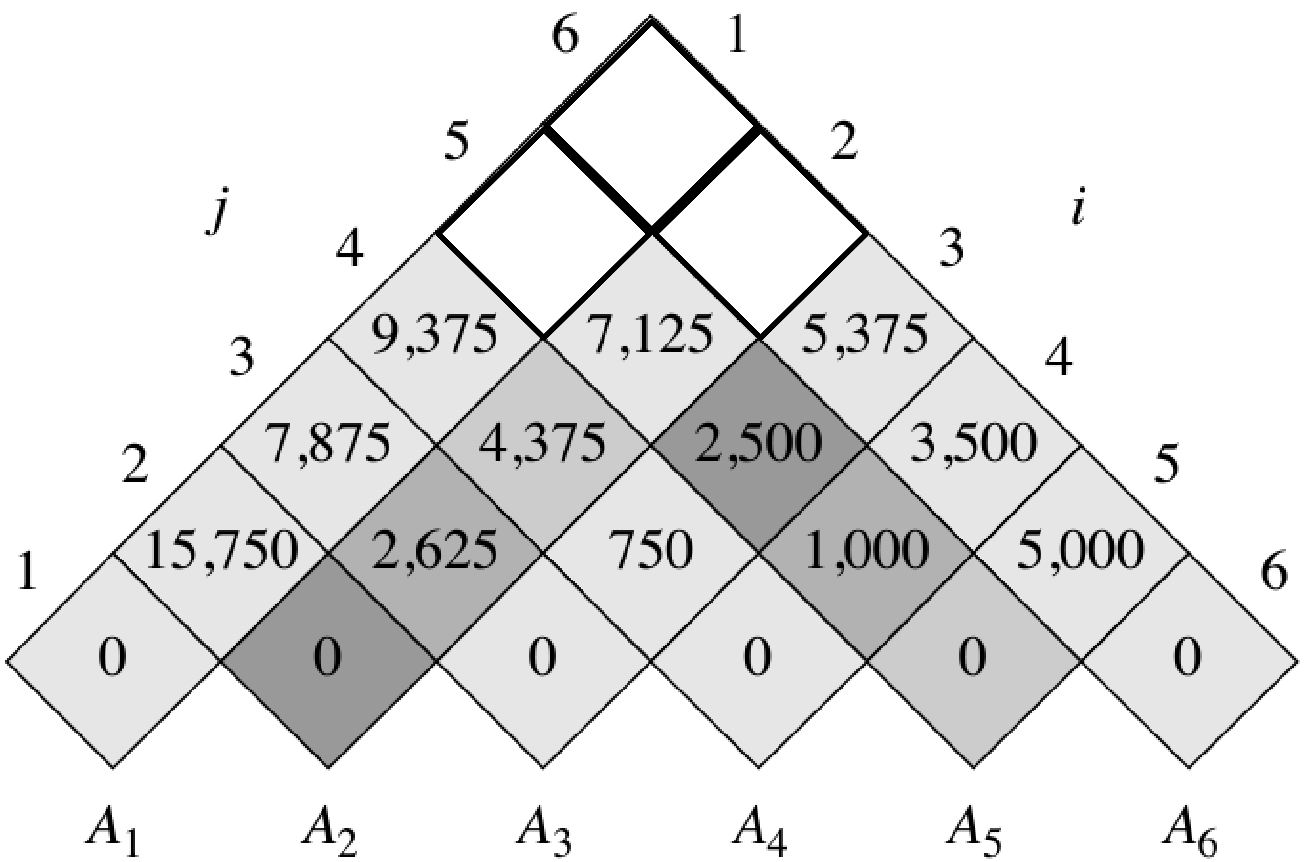

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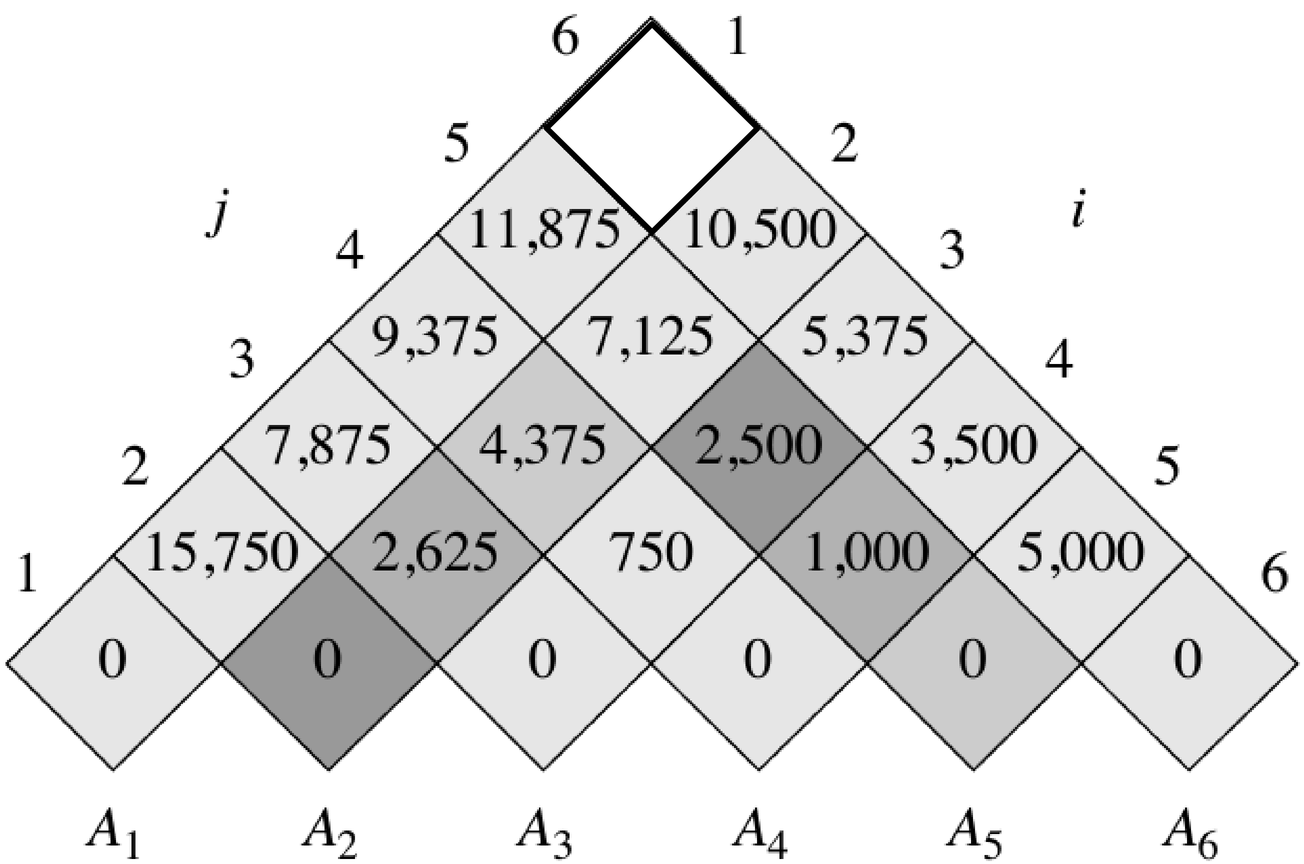

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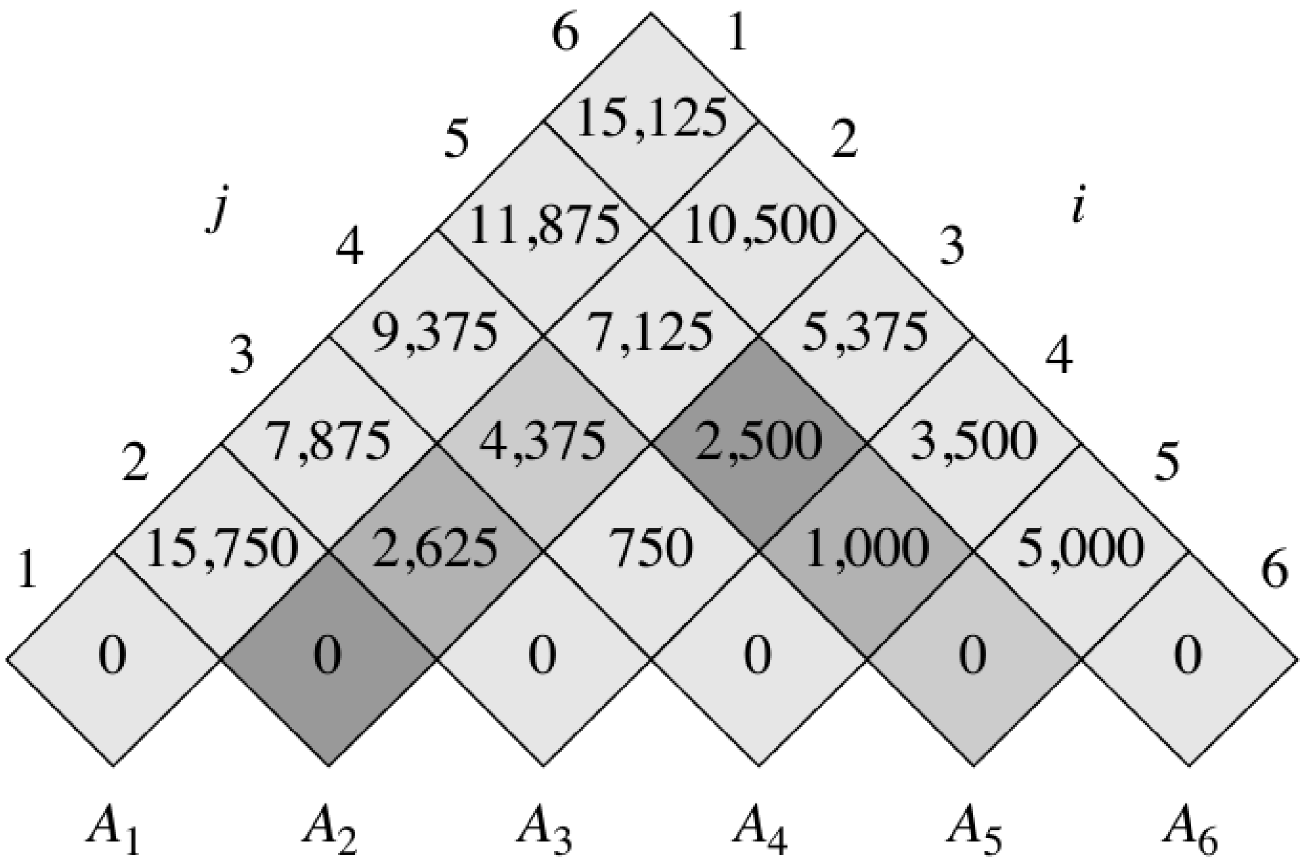

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Matrix Chain Multiplication

```
MATRIX-CHAIN-ORDER(p)
n = p.length − 1
let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0
for l = 2 to n                      // l is the chain length
    for i = 1 to n − l + 1
        j = i + l − 1
        m[i, j] = ∞
        for k = i to j − 1
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q
                s[i, j] = k
return m and s
```

Longest Common Subsequence

- Required reading:
  - Book section 15.4

Book Readings

- Ch. 15: 15.1-15.4