CS/MATH 111, Discrete Structures - Winter 2018. Discussion 8 - Graphs

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Outline

Euler path and tour

Definition 1.1

An *Euler tour* in a graph G is a simple circuit containing **every edge** of G. An *Euler path* in G is a simple path containing every edge of G.

Euler tour

- An Euler tour (or Eulerian tour, Euler circuit) traverses each edge of the graph **exactly once**.
- Graphs that have an Euler tour are called Eulerian.

Necessary and sufficient conditions for Euler circuits and paths

Theorem 1

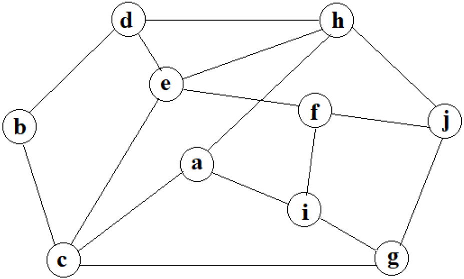
An undirected graph has a closed Euler circuit iff it is connected and each vertex has an even degree.

Theorem 2

An undirected graph has an Euler path but not an Euler tour iff it has exactly two vertices of odd degree.

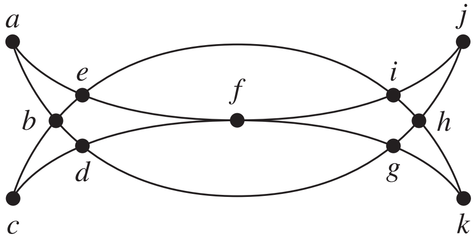
Euler tour

- So this graph is not Eulerian:



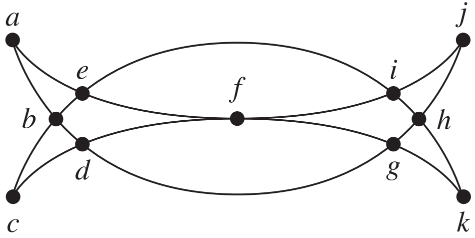
Euler tour

- Mohammed's Scimitars:



Euler tour

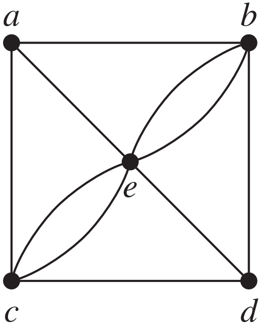
- Mohammed's Scimitars:



a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a

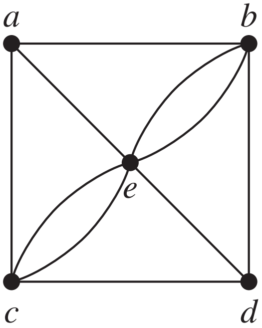
Euler tour

- Determine weather the given graph has an Euler circuit:



Euler tour

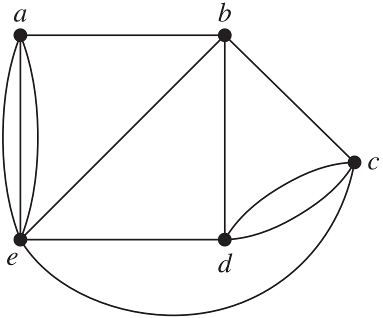
- Determine weather the given graph has an Euler circuit:



No. It has nodes with odd degree...

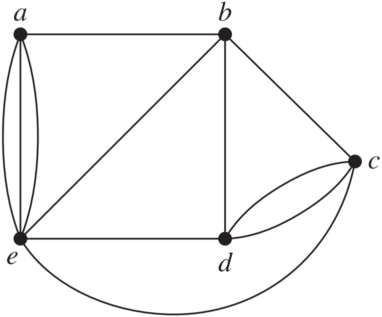
Euler tour

- Determine weather the given graph has an Euler circuit:



Euler tour

- Determine weather the given graph has an Euler circuit[1]:



Yes, all nodes have even degree...

[1] Have a look at Rosen's book [pg.696] for an algorithm...

Outline

Hamiltonian path and circuit

Definition 2.1

A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, \ldots, x_{n1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = x_0, x_1, \ldots, x_{n1}, x_n$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n1}, x_n$ is a Hamilton path.

Hamiltonian Cycle

- Hamiltonian Cycle (or Hamilton circuit/tour) is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once
- Graphs that have an Hamilton tour are called Hamiltonian.

Conditions for the existence of Hamilton circuits

Theorem 3 (Dirac's Theorem)

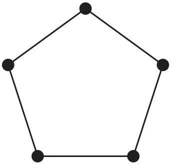
If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamilton cycle.

Theorem 4 (Ore's Theorem)

If G is a simple graph with n vertices with $n \geq 3$ such that $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton cycle.

Conditions for the existence of Hamilton circuits

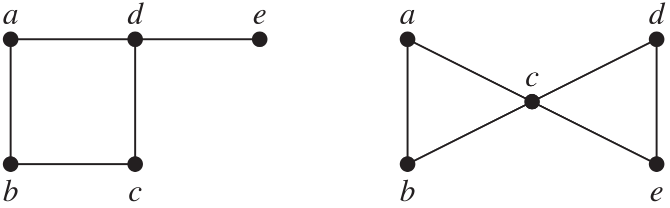
- Both Ore's theorem and Dirac's theorem provide sufficient conditions for a connected simple graph to have a Hamilton circuit.
- However, these theorems do not provide necessary conditions for the existence of a Hamilton circuit.
- For example, the graph C_5 has a Hamilton circuit but does not satisfy the hypotheses of either Ore's theorem or Dirac's theorem.



C_5

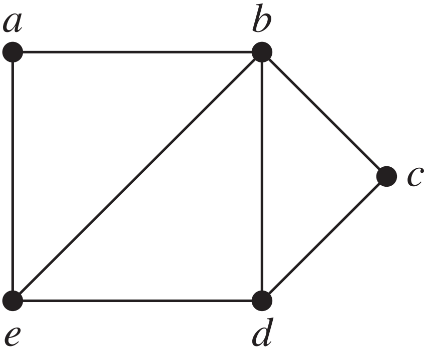
Hamiltonian Cycle

- Show that neither graph displayed has a Hamilton circuit:



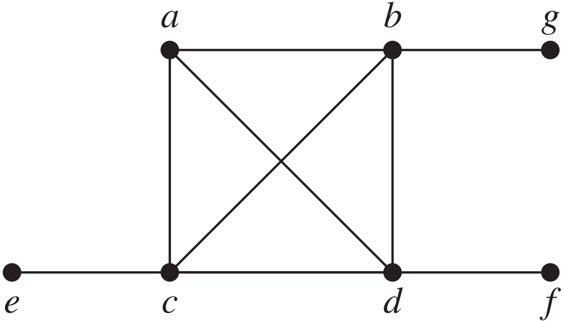
Hamiltonian Cycle

- Determine weather the given graph has a Hamilton circuit:



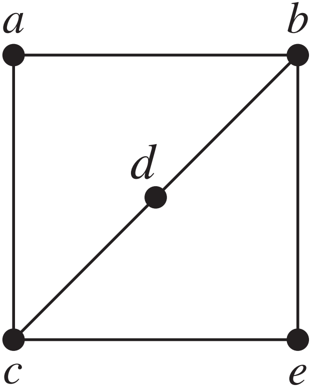
Hamiltonian Cycle

- Determine weather the given graph has a Hamilton circuit:



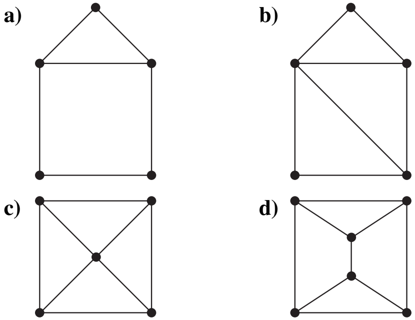
Hamiltonian Cycle

- Determine weather the given graph has a Hamilton circuit:

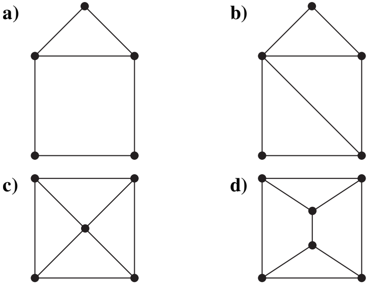


Hamiltonian Cycle

- For each of these graphs, determine:
  - (i) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit
  - (ii) whether Ore's theorem can be used to show that the graph has a Hamilton circuit
  - (iii) whether the graph has a Hamilton circuit



Hamiltonian Cycle

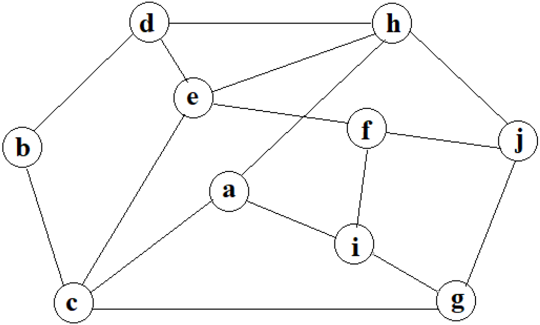


a) No Dirac, No Ore but it has a Hamilton circuit!
b) Same that a).
c) Both Dirac and Ore guarantee the existence of a Hamilton circuit.
d) Same that c).

Although not illustrated in any of the examples in this exercise, there are graphs for which Ore's theorem applies, even though Dirac's does not.

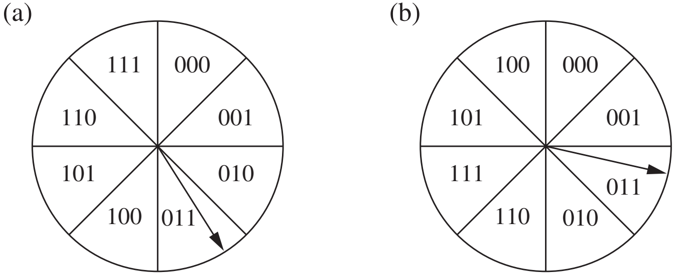
Exercise

- Does this graph have Hamiltonian cycle?



Gray codes [2]

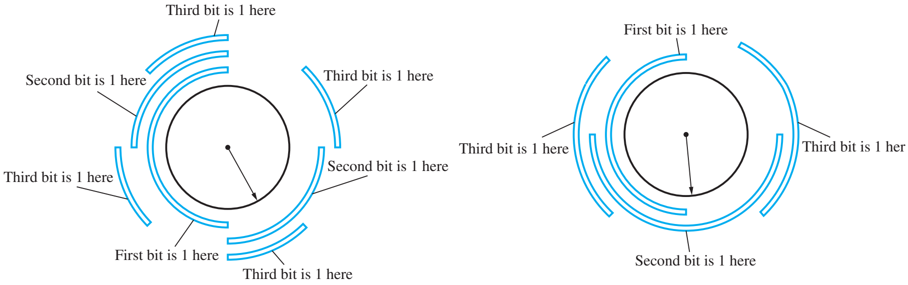
- Converting the position of a pointer into digital form:



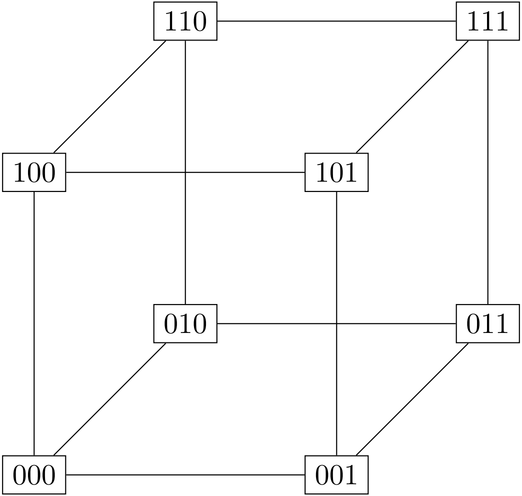
[2] https://en.wikipedia.org/wiki/Gray_code

Gray codes

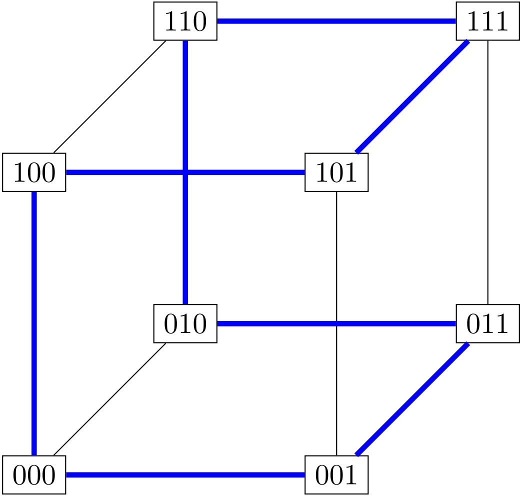
- The digital representation of the position of the pointer:



Gray codes



Gray codes



Outline

Vertex Coloring

Definition 3.1

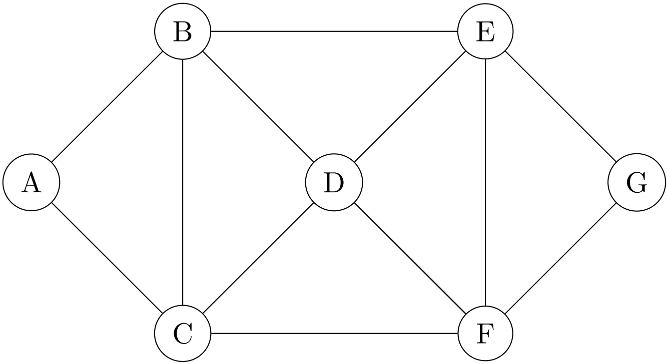
A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Definition 3.2

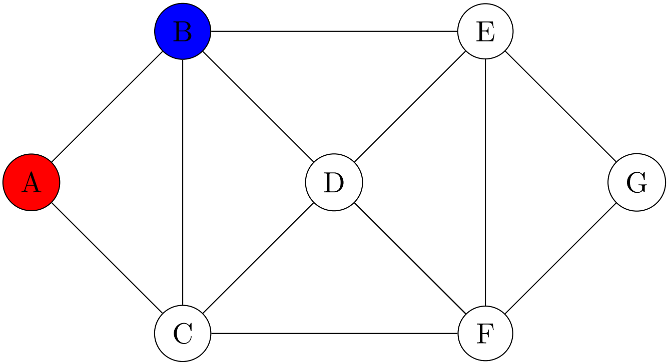
The chromatic number of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by $\chi(G)$. (Here χ is the Greek letter chi.)

A very hard problem(an NP-Complete problem).

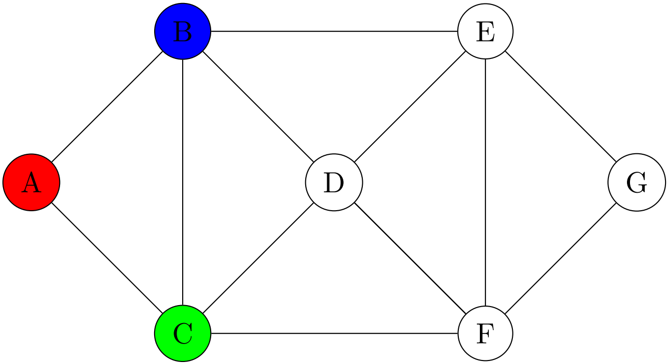
Vertex Coloring



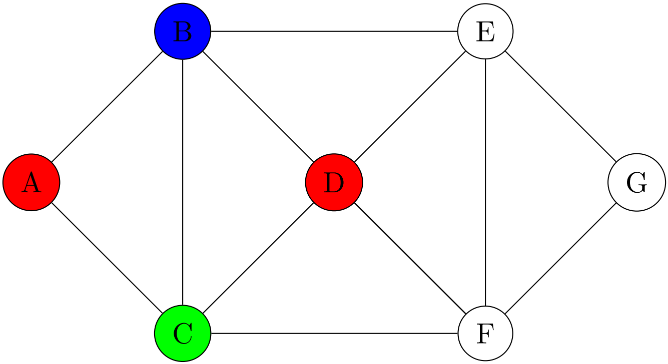
Vertex Coloring



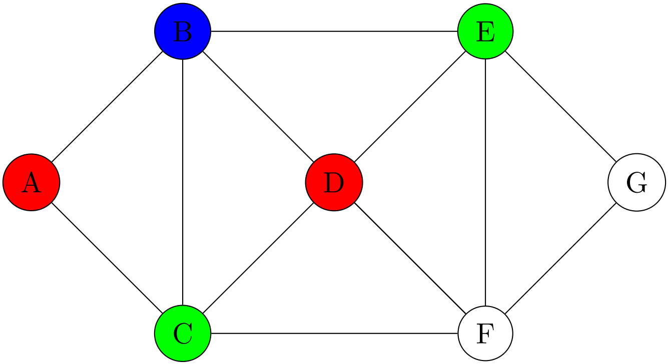
Vertex Coloring



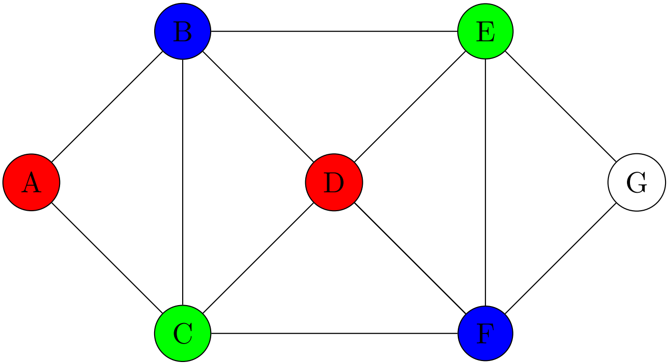
Vertex Coloring



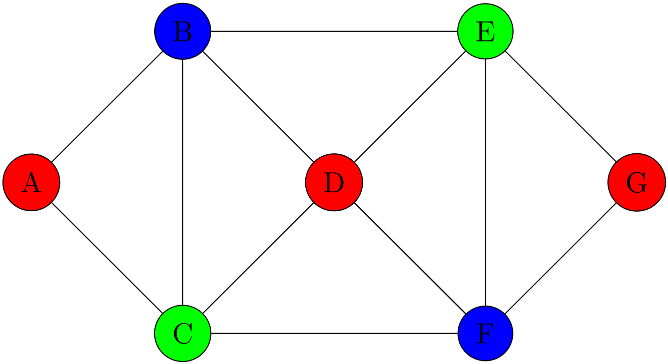
Vertex Coloring



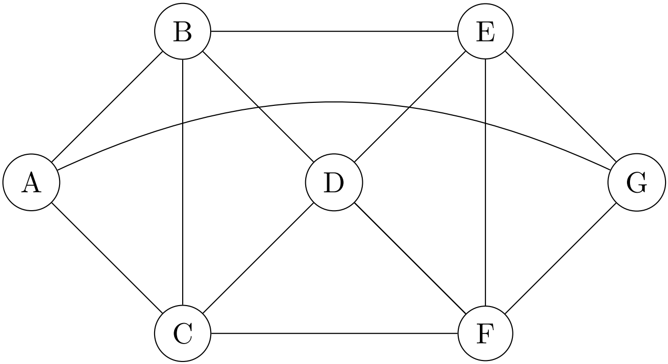
Vertex Coloring



Vertex Coloring



Vertex Coloring



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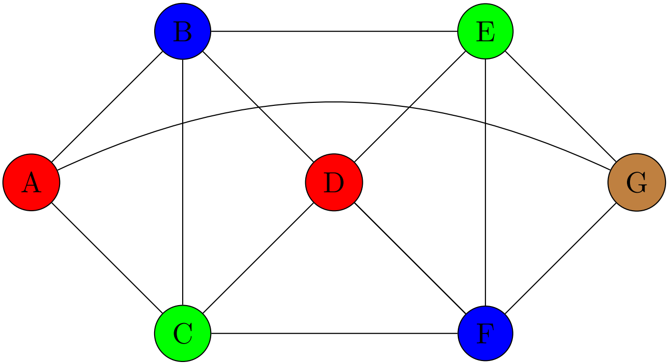

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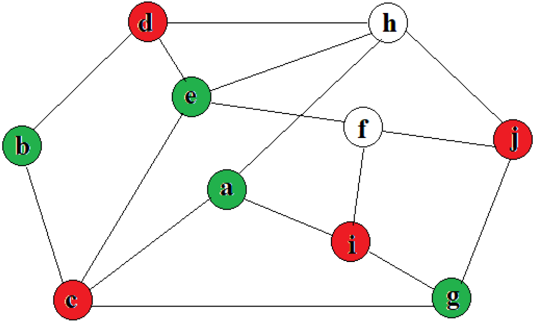
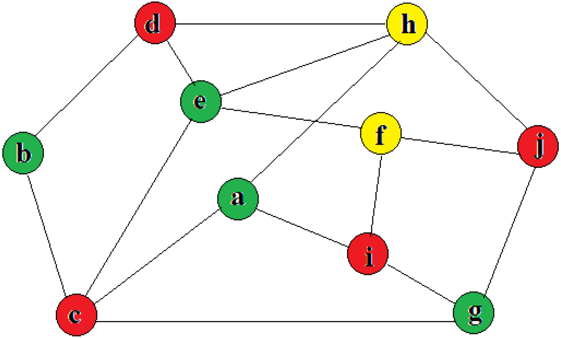
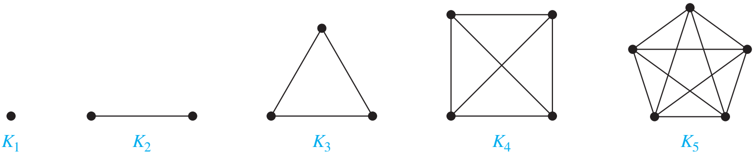


Vertex Coloring



Hamiltonian Cycle

- Complete graphs of n vertices (K_n):



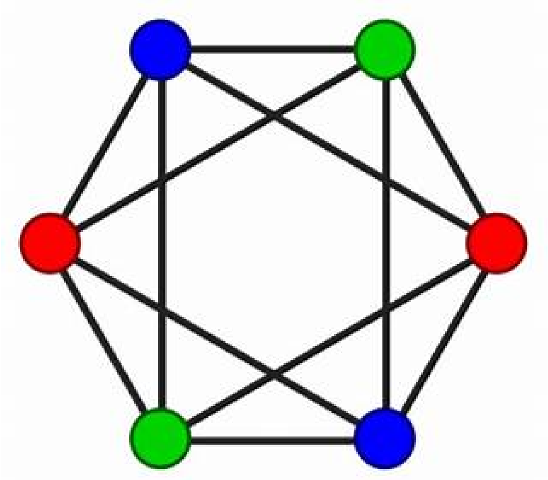
Vertex Coloring

For certain classes of graphs, we can easily compute the chromatic number. For example, the chromatic number of K_n is n, for any n. Notice that we have to argue two separate things to establish that this is its chromatic number:

- K_n can be colored with n colors.
- K_n cannot be colored with less than n colors.

For K_n, both of these facts are fairly obvious. Assigning a different color to each vertex will always result in a well-formed coloring (though it may be a waste of colors). Since each vertex in K_n is adjacent to every other vertex, no two can share a color. So fewer than n colors can't possibly work.

Vertex Coloring



Frequency Assignments

- Television channels 2 through 13 are assigned to stations in Colombia so that no two stations within 150 Km can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Frequency Assignments

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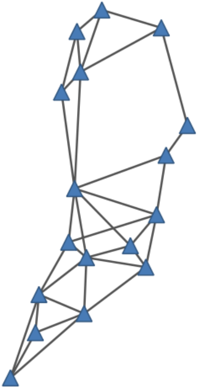

Frequency Assignments

- Television channels 2 through 13 are assigned to stations in Colombia so that no two stations within 150 Km can operate on the same channel. How can the assignment of channels be modeled by graph coloring?
- Construct a graph by assigning a vertex to each station. Two vertices are connected by an edge if they are located within 150 Km of each other. An assignment of channels corresponds to a coloring of the graph, where each color represents a different channel.

Reference

- Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.
  Chapter 10: Graphs.
  Section 10.5: Euler and Hamilton Paths.
  Section 10.8: Graph Coloring.