CS/MATH 111, Discrete Structures - Winter 2019. Discussion 7 - Non-homogeneous Recurrences, Tiling & Red Riding Hood problem

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Outline

Non-homogeneous recurrence[1]

Theorem 1

$$f_n = f_n^{'} + f_n^{''}$$

If $\{f_n^{''}\}$ *is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients:*

$$f_n = c_1 \cdot f_{n-1} + c_2 \cdot f_{n-2} + \cdots + c_k \cdot f_{n-k} + g(n)$$

then every solution is of the form $\{f_n^{'} + f_n^{''}\}$*, where* $\{f_n^{'}\}$ *is a solution of the associated homogeneous recurrence relation.*

[1] Proof available at [Rosen, 2015. pg 521].

Non-homogeneous recurrence

Solve next non-homogeneous recurrence with initial condition $f_0 = 0$, $f_1 = 2$ and $f_2 = 7$:

$$f_n = 6 \cdot f_{n-2} + 4 \cdot f_{n-3} + 2^n \tag{1}$$

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- $f_n^{'} = 6 \cdot f_{n-2} + 4 \cdot f_{n-3}$
  1. Caracteristic equations and its roots:

$$x^3 - 6x - 4 = 0$$

$$(x+2)(x^2 - 2x - 2) = 0$$

$$x_1 = -2,\ x_2 = 1 + \sqrt{3},\ x_3 = 1 - \sqrt{3}$$

  2. General form of the solution:

$$f_n^{'} = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot (1+\sqrt{3})^n + \alpha_3 \cdot (1-\sqrt{3})^n$$

Non-homogeneous recurrence

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$$f_n = 6 \cdot f_{n-2} + 4 \cdot f_{n-3} + 2^n \tag{1}$$

- $g(n) = 2^n$, so:

$$f_n^{''} = p_0 \cdot 2^n \tag{2}$$

- Plug (2) in (1) becomes:

$$p_0 \cdot 2^n = 6 \cdot (p_0 \cdot 2^{n-2}) + 4 \cdot (p_0 \cdot 2^{n-3}) + 2^n$$

$$p_0 = -1 \tag{3}$$

- Finally, (3) in (2):

$$f_n^{''} = -2^n$$

Non-homogeneous recurrence

Solve next non-homogeneous recurrence with initial condition $f_0 = 0$, $f_1 = 2$ and $f_2 = 7$:

$$f_n = 6 \cdot f_{n-2} + 4 \cdot f_{n-3} + 2^n \tag{1}$$

- According to Theorem 1:
  $f_n = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot (1+\sqrt{3})^n + \alpha_3 \cdot (1-\sqrt{3})^n - 2^n$
  3. Initial condition equations and their solutions:
  $$\begin{aligned} f_0 = \quad & \alpha_1 \cdot (-2)^0 + \alpha_2 \cdot (1+\sqrt{3})^0 + \alpha_3 \cdot (1-\sqrt{3})^0 - 2^0 && = 0 \\ f_1 = \quad & \alpha_1 \cdot (-2)^1 + \alpha_2 \cdot (1+\sqrt{3})^1 + \alpha_3 \cdot (1-\sqrt{3})^1 - 2^1 && = 2 \\ f_2 = \quad & \alpha_1 \cdot (-2)^2 + \alpha_2 \cdot (1+\sqrt{3})^2 + \alpha_3 \cdot (1-\sqrt{3})^2 - 2^2 && = 7 \\ & \vdots \end{aligned}$$
  4. Final answer:
  $$\vdots$$

Outline

Example 1[2]

Suppose you are trying to tile a 1 x n walkway with 4 different types of tiles: a red 1x1 tile, a green 1x1 tile, a blue 1x1 tile, and a grey 2x1 tile...

a) Set up and explain a recurrence relation for the number of different tilings for a sidewalk of length n.

b) What is the solution of this recurrence relation?

c) How long must the walkway be in order to have more than 1000 different tiling possibilities?

[2] from `https://tinyurl.com/y6wj64bd`

Example 1

Suppose you have a tiling of length n. This can be built from:

1. a tiling of length $n-1$ followed by a single tile; OR
2. a tiling of length $n-2$ followed by a double tile.

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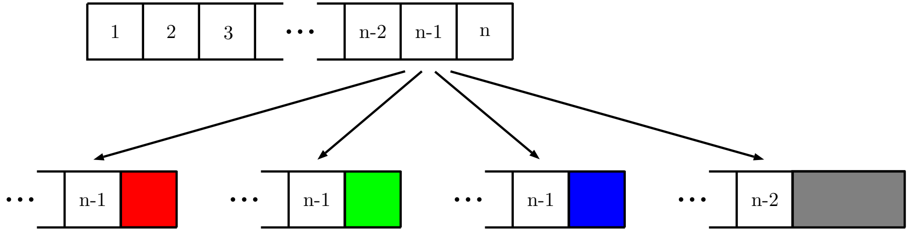
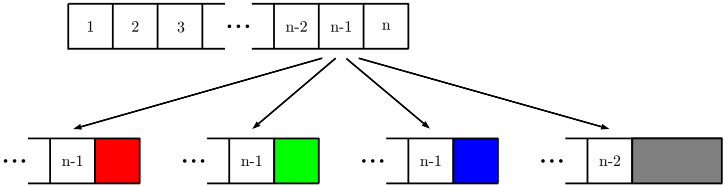


Example 1



- Let T_n be the number of different ways of tiling a 1 x n space. Then for $n \geq 3$:

$$T_n = 3 \cdot T_{n-1} + 1 \cdot T_{n-2} \tag{1}$$

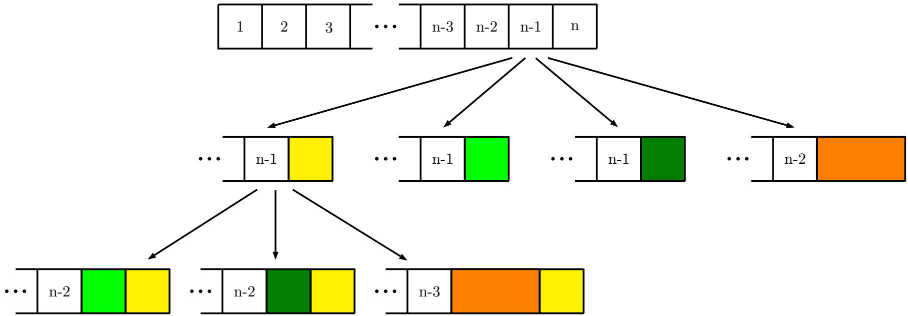
Example 1

- Let T_n be the number of different ways of tiling a 1 x n space. Then for $n \geq 3$:
$$T_n = 3 \cdot T_{n-1} + T_{n-2} \tag{1}$$
- There are 3 possibilities to fill a 1x1 walkway ($n = 1$) and 10 to fill a 2x1 ($n = 2$) walkway, so initial conditions are $T_1 = 3$ and $T_2 = 10$.
- Then by (1):
$$\begin{aligned} T_3 &= 3 \cdot T_2 + T_1 && = 3 \cdot 10 + 3 && = 33 \\ T_4 &= 3 \cdot T_3 + T_2 && = 3 \cdot 33 + 10 && = 109 \\ T_5 &= 3 \cdot T_4 + T_3 && = 3 \cdot 109 + 33 && = 360 \\ T_6 &= 3 \cdot T_5 + T_4 && = 3 \cdot 360 + 109 && = 1189 \end{aligned}$$

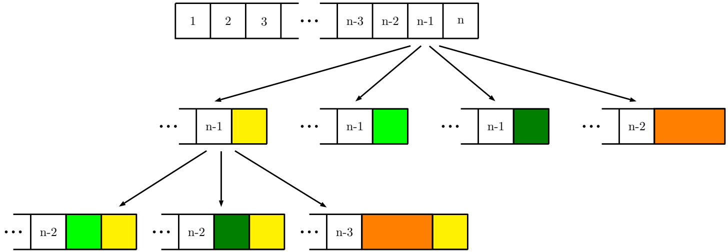
Example 2

We want to tile the n x 1 strip with 2 x 1 and 1 x 1 tiles, using 2 x 1 tiles of orange color and 1 x 1 tiles of three colors: yellow, light-green and dark green. Let T_n be the number of such tilings in which no yellow tiles are next to each other. Determine the formula for T_n be setting up a recurrence equation...

Example 2



Example 2



$$T_n = 2 \cdot T_{n-1} + 3 \cdot T_{n-2} + 1 \cdot T_{n-3}$$

	Initial conditions	
$T_0 =$	Empty tile	$= 1$
$T_1 =$	Y, LG and DG	$= 3$
$T_2 =$	O, LG-Y, DG-Y, ...	$= 9$

Outline

Red Ridding Hood problem

http://www.cs.ucr.edu/~acald013/public/tmp/rrh.pdf