15. Prove Corollary 3.

16. Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \le 2v - 4$ if $v \ge 3$.

***17.** Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \le (5/3)v - (10/3)$ if $v \ge 4$.

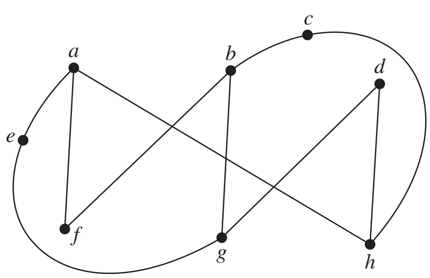
18. Suppose that a planar graph has k connected components, e edges, and v vertices. Also suppose that the plane is divided into r regions by a planar representation of the graph. Find a formula for r in terms of e, v, and k.

19. Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?

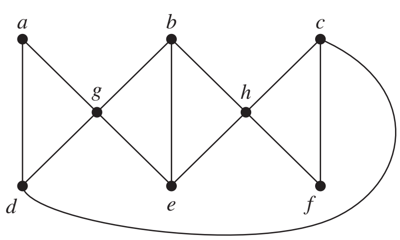
a) K_5 **b)** K_6 **c)** $K_{3,3}$ **d)** $K_{3,4}$

In Exercises 20–22 determine whether the given graph is homeomorphic to $K_{3,3}$.

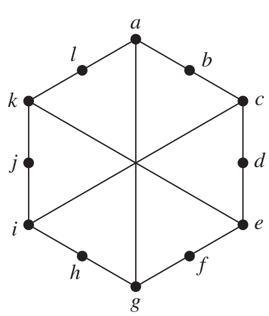
20.



21.

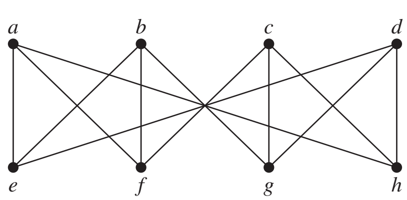


22.

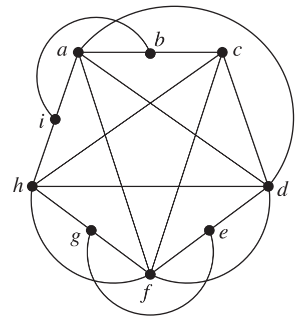


In Exercises 23–25 use Kuratowski's theorem to determine whether the given graph is planar.

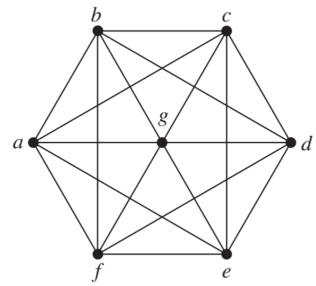
23.



24.



25.



The **crossing number** of a simple graph is the minimum number of crossings that can occur when this graph is drawn in the plane where no three arcs representing edges are permitted to cross at the same point.

26. Show that $K_{3,3}$ has 1 as its crossing number.

****27.** Find the crossing numbers of each of these nonplanar graphs.

a) K_5 **b)** K_6 **c)** K_7
d) $K_{3,4}$ **e)** $K_{4,4}$ **f)** $K_{5,5}$

***28.** Find the crossing number of the Petersen graph.

****29.** Show that if m and n are even positive integers, the crossing number of $K_{m,n}$ is less than or equal to $mn(m-2)(n-2)/16$. [*Hint:* Place m vertices along the x-axis so that they are equally spaced and symmetric about the origin and place n vertices along the y-axis so that they are equally spaced and symmetric about the origin. Now connect each of the m vertices on the x-axis to each of the vertices on the y-axis and count the crossings.]

The **thickness** of a simple graph G is the smallest number of planar subgraphs of G that have G as their union.

30. Show that $K_{3,3}$ has 2 as its thickness.

***31.** Find the thickness of the graphs in Exercise 27.

32. Show that if G is a connected simple graph with v vertices and e edges, where $v \ge 3$, then the thickness of G is at least $\lceil e/(3v-6) \rceil$.

***33.** Use Exercise 32 to show that the thickness of K_n is at least $\lfloor (n+7)/6 \rfloor$ whenever n is a positive integer.

34. Show that if G is a connected simple graph with v vertices and e edges, where $v \ge 3$, and no circuits of length three, then the thickness of G is at least $\lceil e/(2v-4) \rceil$.

35. Use Exercise 34 to show that the thickness of $K_{m,n}$, where m and n are not both 1, is at least $\lceil mn/(2m+2n-4) \rceil$ whenever m and n are positive integers.

***36.** Draw K_5 on the surface of a torus (a doughnut-shaped solid) so that no edges cross.

***37.** Draw $K_{3,3}$ on the surface of a torus so that no edges cross.