CS/MATH 111, Discrete Structures - Winter 2019. Discussion 9 - Graphs and Tree introduction

Andres, Sara, Elena

University of California, Riverside

March 11, 2019

Outline

Bipartite graph

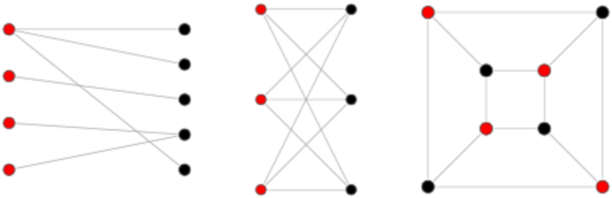
Definition 1.1

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

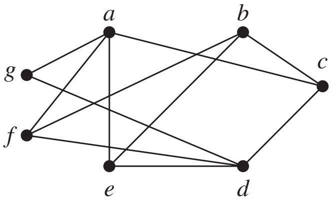
Bipartite graph

- Bipartite graphs are equivalent to two-colorable graphs.
- All acyclic graphs are bipartite.
- A cyclic graph is bipartite iff all its cycles are of even length.

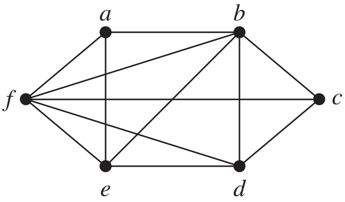
Bipartite graph



Examples

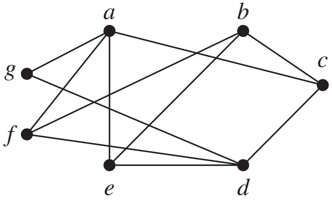


G

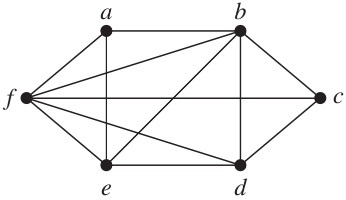


H

Examples



G



H

G: Yes (See $\{a, b, d\}$ and $\{c, e, f, g\}$).

H: No (See $\{a, b, f\}$)

Examples

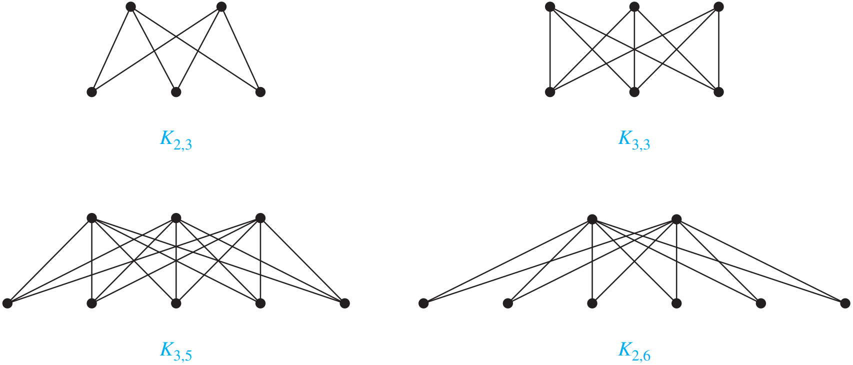


Figure: Complete Bipartite Graphs.

Outline

Perfect matching

- A perfect matching of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.
- A perfect matching is therefore a matching containing $\frac{n}{2}$ edges (the largest possible)[1], meaning perfect matchings are only possible on graphs with an even number of vertices.

[1] http://mathworld.wolfram.com/PerfectMatching.html

Examples

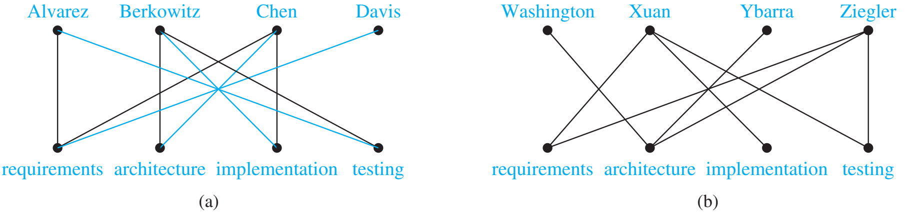


Figure: Modeling Job Assignments for Which Employees Have Been Trained.

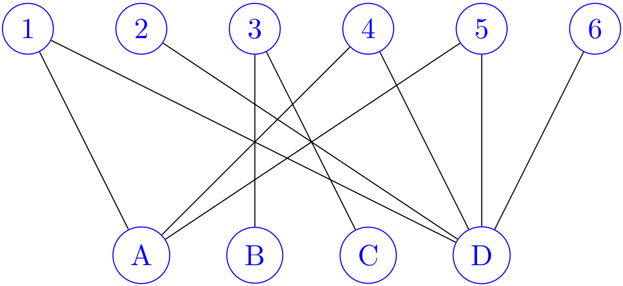
Hall's Theorem[2]

Theorem 1

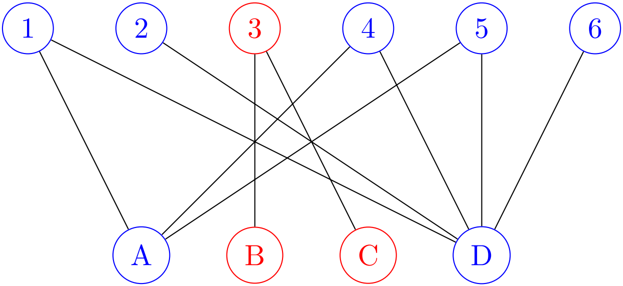
Let $G = (X, Y)$ be a bipartite graph. Then X has a perfect macthing into Y iif for all $T \subseteq X$, the inequality $|T| \leq |N(T)|$ holds. Where $N(T)$ is the set of all neighbors of the vertices in T. In other words, $y \in Y$ is an element of $N(T)$ iif there is a vertex $x \in T$ so that (x, y) is an edge.

[2] Proof available at [Rosen, 2015. pg 660].

Example

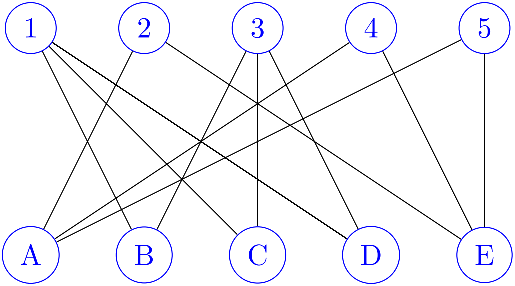


Example

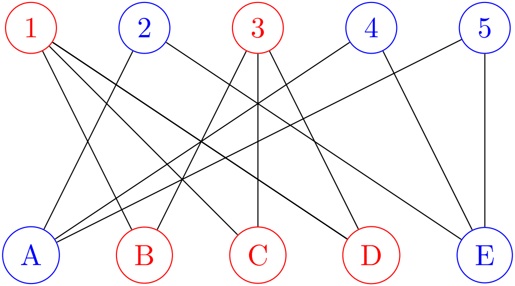


Let $T = \{B, C\}$, $\quad N(T) = \{3\}$, $\quad |T| = 2$ and $|N(T)| = 1$.
Violates Hall's theorem.

Example



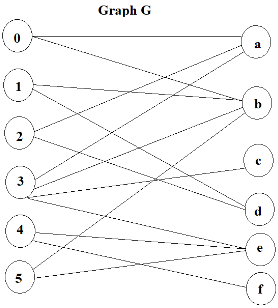
Example



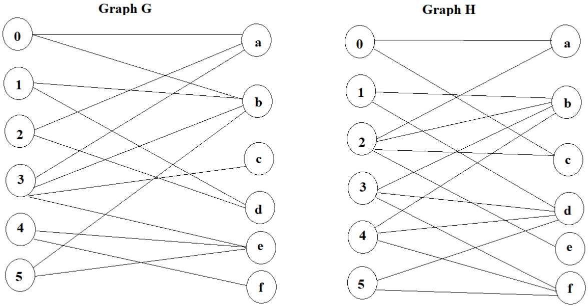
Let $T = \{B, C, D\}$, $N(T) = \{1, 3\}$, $|T| = 3$ and $|N(T)| = 2$.
Violates Hall's theorem.

Example

You are given two bipartite graph G and H below. For each graph determine whether it has a perfect matching. Justify your answer, either by listing the edges that are in the matching or use Hall's Theorem to show that the graph does not have a perfect matching.



Example



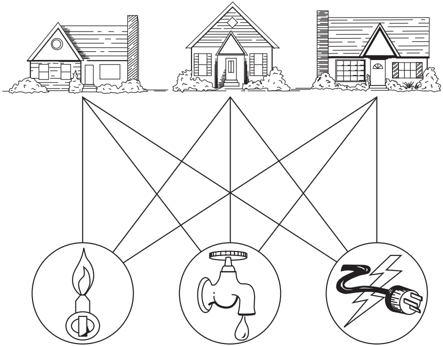
G: Yes, see $\{0, a\}$,$\{1, b\}$,$\{2, d\}$,$\{3, c\}$,$\{4, f\}$ and $\{5, e\}$.

H: No, Let $T = \{a, c, e\}$, then $N(T) = \{0, 2\}$, therefore $|T| \nleq |N(T)|$ which violates Hall's theorem.

Outline

Planar graphs

Is it possible to join these houses and utilities so that none of the connections cross?



Planar graphs

Definition 3.1

A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

Examples

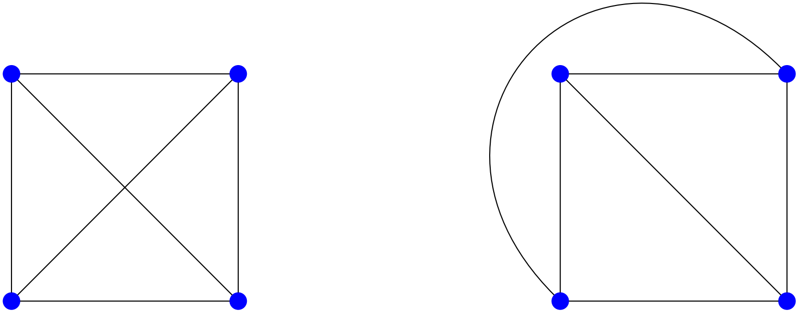


Figure: The K_4 graph and its drawn with no crossings.

Examples

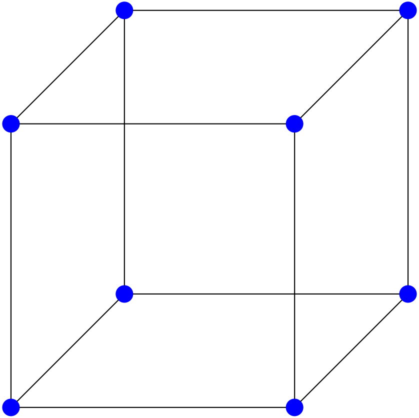


Figure: A Q_3 graph.

Examples

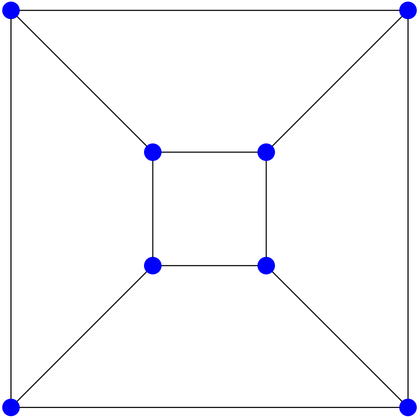


Figure: The planar representation of a Q_3 graph.

Euler's Formula

- A planar representation of a graph splits the plane into regions[3] (including an unbounded region.)
- Euler showed that all planar representations of a graph split the plane into the same number of regions.
- There is a relationship between the number of regions, vertices and edges.

[3] regions = faces.

Euler's Formula

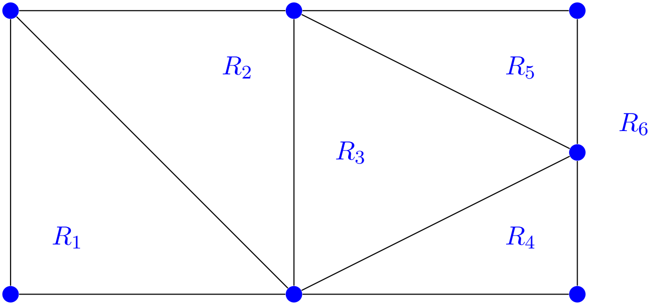


Figure: The Regions of the Planar Representation of a Graph.

Euler's Formula

Theorem 2

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Euler's Formula

Corollary 3

If G is a connected planar simple graph with m edges and n vertices, and $n \geq 3$ and no circuits of length 3, then $m \leq 2n - 4$.

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[4].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[5].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

■

[4] no multiple edges, no loops and no simple cycles of length 3

[5] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[4].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[5].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

∎

[4] no multiple edges, no loops and no simple cycles of length 3

[5] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[4].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[5].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

[4] no multiple edges, no loops and no simple cycles of length 3

[5] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[4].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[5].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

∎

[4] no multiple edges, no loops and no simple cycles of length 3

[5] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[4].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[5].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

∎

[4] no multiple edges, no loops and no simple cycles of length 3

[5] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[4].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[5].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

$\blacksquare$

[4] no multiple edges, no loops and no simple cycles of length 3

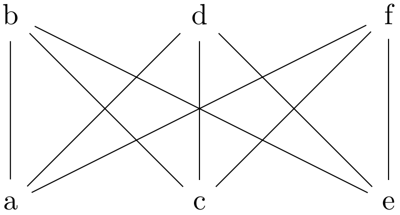
[5] because each edge occurs on the boundary of a region exactly twice

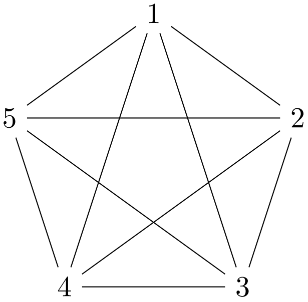
Outline

Kuratowski's theorem

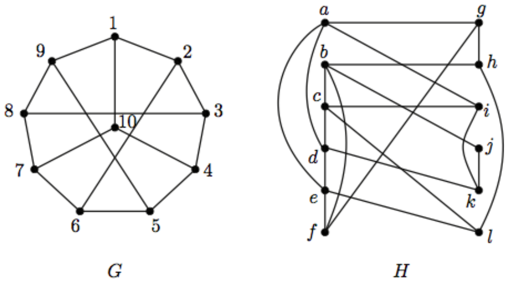
Theorem 4

A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.



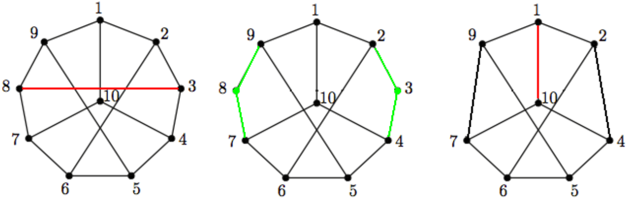


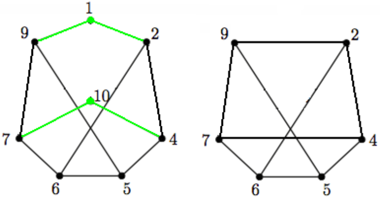
Some examples[6]



6 Taken from `https://tinyurl.com/yyd5cq8g`

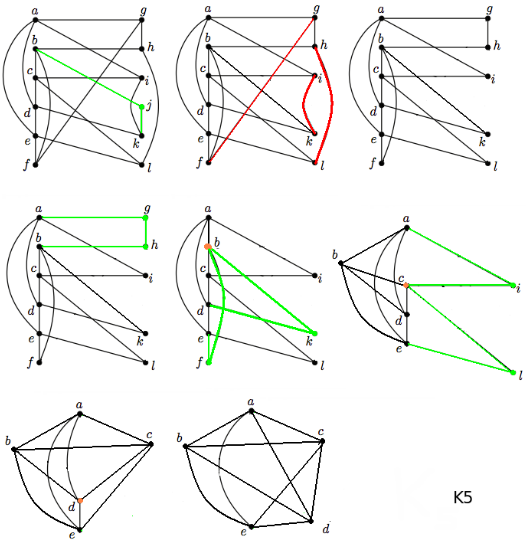
Some examples



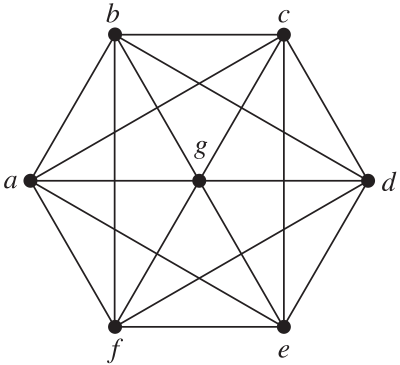


K3,3

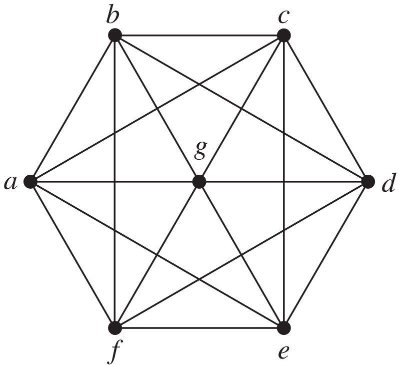
Some examples



Example $K_{3,3}$

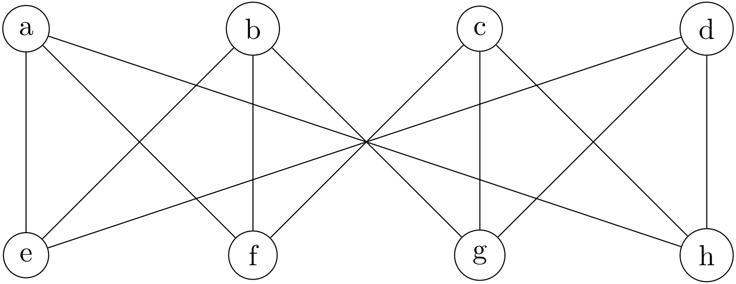


Example $K_{3,3}$

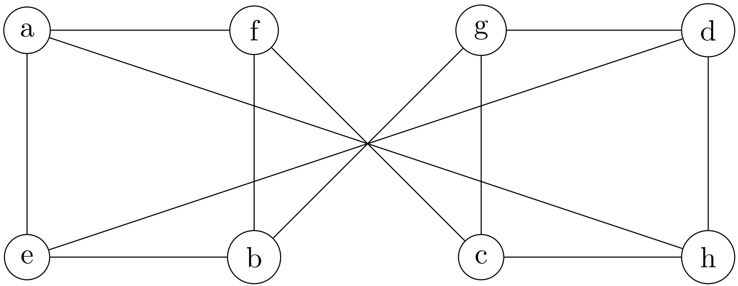


This graph is nonplanar, since it contains $K_{3,3}$ as a subgraph: the parts are $\{a, g, d\}$ and $\{b, c, e\}$.

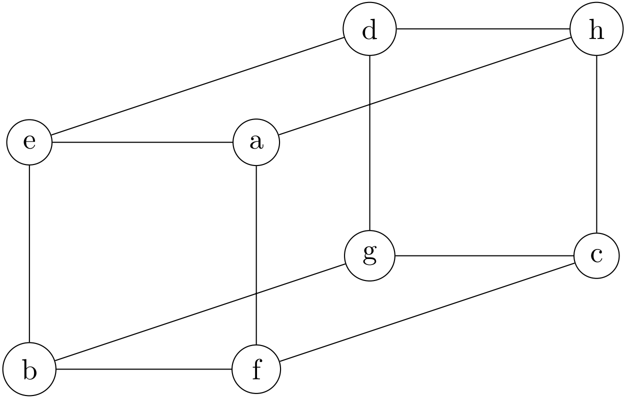
Example Q_3



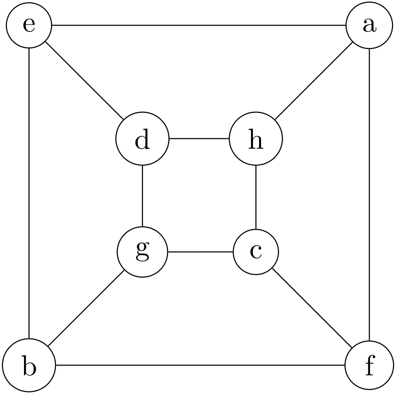
Example Q_3



Example Q_3



Example Q_3



Outline

Trees

Lemma 5

If T is a tree, and has n vertices, then its number of edges is $m = n - 1$.

Taken from Rosen's book (pg 752).

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

■

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $m = n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with $n = k$ vertices has $m = k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $n = k + 1$ vertices, we want to prove that T has k edges.
   - Let's suppose that v is a leaf[7] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[8].
   - By the assumption hypothesis, as T' has k vertices, it has $k - 1$ edges.
   - It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

■

[7] It must exist because the tree is finite

[8] T' is still connected and has no simple circuits.

Reference

- Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.
  - Chapter 10. Graphs:
    Section 10.2: Graph Terminology and Special Types of Graphs.
    Section 10.7: Planar Graphs.
  - Chapter 11. Trees:
    Section 11.1: Introduction to Trees.