Little Red Riding Hood

Little Red Riding Hood is assembling a fruit basket for her sick grandmother. The basket will contain 15 fruit, including apples, bananas, mangos, and strawberries (and no other fruit). The basket must contain:

- at least 2 but no more than 6 apples, **and**
- no more than 4 bananas, **and**
- at least 1 but no more than 5 mango, **and**
- no more than 3 strawberries.

Determine the number of ways to assemble the fruit basket.

Suppose we have

x apples

y bananas

z mangos

d strawberries

Then as the problem states

$$x + y + z + d = 15$$

Using the conditions we can get

$$\begin{cases} 2 \leq x \leq 6 \\ 0 \leq y \leq 4 \\ 1 \leq z \leq 5 \\ 0 \leq d \leq 3 \end{cases}$$

Let’s make the left sides of all inequalities 0

Let x’= x – 2 then

$$2 \leq x' + 2 \leq 6$$
$$0 \leq x' \leq 4$$

Similarly, let z’= z – 1 then

$$1 \leq z' + 1 \leq 5$$
$$0 \leq z' \leq 4$$

Now we have

$$x' + y + z' + d = 12$$

$$\begin{cases} 0 \leq x' \leq 4 \\ 0 \leq y \leq 4 \\ 0 \leq z' \leq 4 \\ 0 \leq d \leq 3 \end{cases}$$

We need to determine the number of ways to assemble the fruit basket

The number of ways to assemble the fruit basket is

$$S(x' \leq 4 \ \cap y \ \leq 4 \ \cap z' \leq 4 \ \cap d \ \leq 3) =$$

$$= S - S(x' \geq 5 \cup y \ \geq 5 \cup z' \geq 5 \cup d \ \geq 4)$$

where S is the number of ways to assemble the basket without having constraints on the variables.

In general

- If there are no constraints on the variables then
$$S = \binom{m+k-1}{k-1}$$
- When there are lower bounds on the variables ($a_i \le x_i, a_i \ge 0$), then the number of solutions is
$$\binom{m-A+k-1}{k-1}$$

where $A = \sum a_i$

In our case m=15, k=4 (we have 4 variables: x',y,z',d)

$$S = \binom{12+4-1}{4-1} = \frac{15!}{(15-3)!\,3!} = \frac{13 \cdot 14 \cdot 15}{6}$$
$$= 455$$

So now we have

$$S(x' \leq 4 \ \cap y \ \leq 4 \ \cap z' \leq 4 \ \cap d \ \leq 3) =$$

$$= 455 - S(x' \geq 5 \cup y \ \geq 5 \cup z' \geq 5 \cup d \ \geq 4)$$

From the Inclusion-Exclusion we have

$$S(x' \geq 5 \cup y \ \geq 5 \cup z' \geq 5 \cup d \ \geq 4) =$$
$$= S(x' \geq 5) + \ S(y \geq 5) + S(z' \geq 5) + S(d \geq 4)$$
$$- S(x' \geq 5 \cap y \geq 5) - S(x' \geq 5 \cap z' \geq 5)$$
$$- S(x' \geq 5 \cap d \geq 4) - S(y \geq 5 \cap z' \geq 5)$$
$$- S(y \geq 5 \cap d \geq 4) - S(z' \geq 5 \cap d \geq 4)$$
$$+ S(x' \geq 5 \cap y \geq 5 \cap z' \geq 5) + S(x' \geq 5 \cap z' \geq 5 \cap d \geq 4)$$
$$+ S(x' \geq 5 \cap y \geq 5 \cap d \geq 4)$$
$$+ S(y \geq 5 \cap z' \geq 5 \cap d \geq 4)$$
$$- S(x' \geq 5 \cap y \geq 5 \cap z' \geq 5 \cap d \geq 4)$$

Using the information in the slide 6 we can compute:

$$S(x' \geq 5) = S(y \geq 5) = S(z' \geq 5)$$

$$= \binom{12 - 5 + 4 - 1}{4 - 1} = \frac{10!}{(10 - 3)!\,3!} = \frac{8 \cdot 9 \cdot 10}{6}$$

$$= 120$$

$$S(d \geq 4) = \binom{12 - 4 + 4 - 1}{4 - 1} = \frac{11!}{(11 - 3)!\,3!}$$

$$= \frac{9 \cdot 10 \cdot 11}{6} = 165$$

Using the information in the slide 6 we can compute:

$$S(x' \geq 5 \cap y \geq 5) = S(x' \geq 5 \cap z' \geq 5) = S(y \geq 5 \cap z' \geq 5)$$
$$= \binom{12 - 10 + 4 - 1}{4 - 1} = \frac{5!}{(5-3)!\,3!} = \frac{3 \cdot 4 \cdot 5}{6} = 10$$

$$S(x' \geq 5 \cap d \geq 4) = S(y \geq 5 \cap d \geq 4) = S(z' \geq 5 \cap d \geq 4)$$
$$= \binom{12 - 9 + 4 - 1}{4 - 1} = \frac{6!}{(6-3)!\,3!} = \frac{4 \cdot 5 \cdot 6}{6} = 20$$

Using the information in the slide 6 we can compute:

$$S(x' \geq 5 \cap y \geq 5 \cap z' \geq 5) = \binom{12 - 15 + 4 - 1}{4 - 1} = \binom{0}{3} = 0$$

$$S(x' \geq 5 \cap z' \geq 5 \cap d \geq 4) = S(x' \geq 5 \cap y \geq 5 \cap d \geq 4)$$
$$= S(y \geq 5 \cap z' \geq 5 \cap d \geq 4) = \binom{12 - 14 + 4 - 1}{4 - 1} = \binom{1}{3}$$
$$= 0$$

Using the information in the slide 6 we can compute:

$$S(x' \geq 5 \cap y \geq 5 \cap z' \geq 5 \cap d \geq 4)$$
$$= \binom{12 - 19 + 4 - 1}{4 - 1} = 0$$

Finally

$$S(x' \leq 4\ \cap y\ \leq 4\ \cap z' \leq 4\ \cap d\ \leq 3)$$
$$= 455$$
$$- (3 \cdot 120 + 165 - 3 \cdot 10 - 3 \cdot 20 + 0 + 0$$
$$- 0) = 455 - (360 + 165 - 30 - 60)$$
$$= 455 - 435 = 20$$