

# CS/MATH 111, Discrete Structures - Winter 2019.

## Discussion 3 - Modular Arithmetic

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# Outline

# Definition

$$\frac{A}{B} = Q \text{ remainder } R$$

- ▶ A is the dividend
- ▶ B is the divisor
- ▶ Q is the quotient
- ▶ R is the remainder

$$A \bmod B = R$$

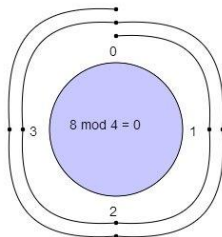
# Definition

$$\frac{13}{5} = 2 \text{ remainder } 3$$

$$13 \bmod 5 = 3$$

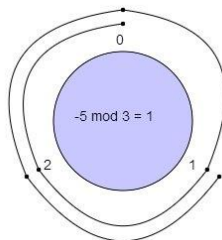
# Definition

$$8 \bmod 4 = ?$$



# Definition

$$-5 \bmod 3 = ?$$



# Definition

$$A \bmod B = (A + K \cdot B) \bmod B$$

For example:

$$3 \bmod 10 = 3$$

$$13 \bmod 10 = 3$$

$$23 \bmod 10 = 3$$

$$33 \bmod 10 = 3$$

# Congruence modulo

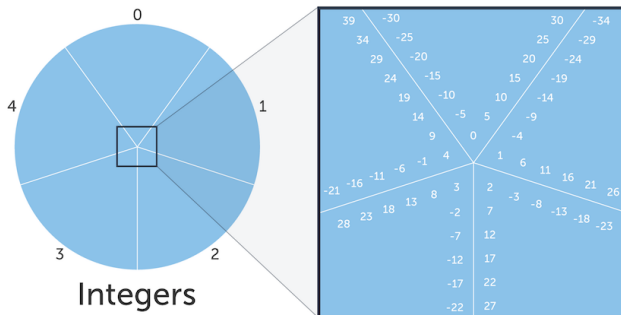
Congruence modulo:

$$A \equiv B \pmod{C}$$



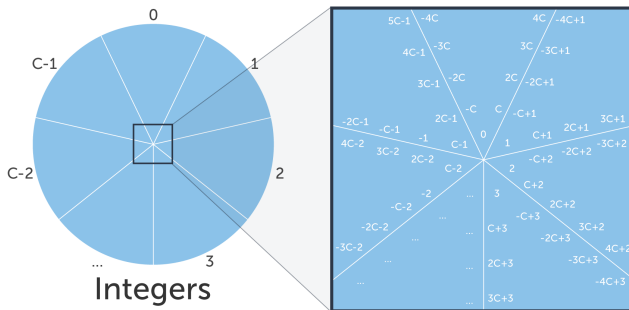
# Congruence modulo

$$26 \equiv 11 \pmod{5}$$



# Congruence modulo

$$A \equiv B \pmod{C}$$



# Equivalent Statements

Equivalent Statements:

- ▶  $A \equiv B \pmod{C}$
- ▶  $A \bmod C = B \bmod C$
- ▶  $C \mid (A - B)$
- ▶  $A = B + K \cdot C$

# Equivalent Statements

For example:

- ▶  $13 \equiv 23 \pmod{5}$
- ▶  $13 \bmod 5 = 23 \bmod 5$
- ▶  $5 \mid (13 - 23)$  by  $5 \times -2 = -10$
- ▶  $13 = 23 + K \cdot 5$  by  $K = -2$

# Equivalence relation

Equivalence relation:

- ▶  $A \equiv A \pmod{C}$  [**reflexive**]
- ▶  $A \equiv B \pmod{C}$  then  $B \equiv A \pmod{C}$  [**symmetric**]
- ▶  $A \equiv B \pmod{C}$  and  $B \equiv D \pmod{C}$  then  $A \equiv D \pmod{C}$  [**transitive**]

# Equivalence relation

For example:

- ▶  $3 \equiv 3 \pmod{5}$
- ▶  $3 \equiv 8 \pmod{5}$  then  $8 \equiv 3 \pmod{5}$
- ▶  $3 \equiv 8 \pmod{5}$  and if  $8 \equiv 18 \pmod{5}$  then  $3 \equiv 18 \pmod{5}$

# Outline

# The quotient remainder theorem

Given any integer  $A$ , and a **positive** integer  $B$ , there exist unique integers  $Q$  and  $R$  such that:

$$A = B * Q + R \text{ where } 0 \leq R < B$$

If we can write a number in this form then

$$A \bmod B = R$$




# Modular Addition and Subtraction<sup>1</sup>

$$(A + B) \bmod C = (A \bmod C + B \bmod C) \bmod C$$

Example:

$$A = 14, B = 17, C = 5$$

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<sup>1</sup>For prove have a look at <https://tinyurl.com/yaltbzgza> 

# Modular Addition and Subtraction

Solve for Y:

$$(699 + 997) \bmod 3 = Y$$

# Modular Addition and Subtraction

Solve for Y:

$$(699 + 997) \bmod 3 = Y$$

$$Y = 1$$

# Modular Addition and Subtraction

Give:

$$A \bmod 8 = 3$$

$$(A + 19) \bmod 8 = Y$$

Solve for Y.

# Modular Addition and Subtraction

Give:

$$A \bmod 8 = 3$$

$$(A + 19) \bmod 8 = Y$$

Solve for Y.

$$Y = 6$$

# Outline


# Modular multiplication<sup>2</sup>

$$(A * B) \bmod C = (A \bmod C * B \bmod C) \bmod C$$

Example:

$$A = 4, B = 7, C = 6$$

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<sup>2</sup>For prove have a look at <https://tinyurl.com/ksf5muz> 

# Modular exponentiation

$$A^B \bmod C = ((A \bmod C)^B) \bmod C$$

Example:

Let's solve:

$$2^{90} \bmod 13$$

but we have a calculator that can't hold any numbers larger than  $2^{50}$ ...



# Modular exponentiation

Example:

$$2^{90} \bmod 13 = 2^{50} * 2^{40} \bmod 13$$

$$2^{90} \bmod 13 = (2^{50} \bmod 13 * 2^{40} \bmod 13) \bmod 13$$

$$2^{90} \bmod 13 = (2^{50} \bmod 13 * 2^{40} \bmod 13) \bmod 13$$

Using our calculator we know:

$$2^{50} \bmod 13 = 1125899906842624 \bmod 13 = 4$$

$$2^{40} \bmod 13 = 1099511627776 \bmod 13 = 3$$

# Modular exponentiation

Example:

$$2^{90} \bmod 13 = 2^{50} * 2^{40} \bmod 13$$

$$2^{90} \bmod 13 = (2^{50} \bmod 13 * 2^{40} \bmod 13) \bmod 13$$

$$2^{90} \bmod 13 = (2^{50} \bmod 13 * 2^{40} \bmod 13) \bmod 13$$

$$2^{90} \bmod 13 = (4 * 3) \bmod 13$$

$$2^{90} \bmod 13 = 12 \bmod 13$$

$$2^{90} \bmod 13 = 12$$

## Modular exponentiation by squaring

$$A^2 \bmod C = (A * A) \bmod C = ((A \bmod C) * (A \bmod C)) \bmod C$$

Solve:

$$7^{256} \bmod 13$$

# Modular exponentiation by squaring

$$7^1 \bmod 13 = 7$$

$$7^2 \bmod 13 = (7^1 * 7^1) \bmod 13 = (7^1 \bmod 13 * 7^1 \bmod 13) \bmod 13$$

## Modular exponentiation by squaring

$$7^1 \bmod 13 = 7$$

$$7^2 \bmod 13 = (7^1 * 7^1) \bmod 13 = (7^1 \bmod 13 * 7^1 \bmod 13) \bmod 13$$

$$7^2 \bmod 13 = (7 * 7) \bmod 13 = 49 \bmod 13 = 10$$

$$7^2 \bmod 13 = 10$$

$$7^4 \bmod 13 = (7^2 * 7^2) \bmod 13 = (7^2 \bmod 13 * 7^2 \bmod 13) \bmod 13$$

## Modular exponentiation by squaring

$$7^1 \bmod 13 = 7$$

$$7^2 \bmod 13 = (7^1 * 7^1) \bmod 13 = (7^1 \bmod 13 * 7^1 \bmod 13) \bmod 13$$

$$7^2 \bmod 13 = (7 * 7) \bmod 13 = 49 \bmod 13 = 10$$

$$7^2 \bmod 13 = 10$$

$$7^4 \bmod 13 = (7^2 * 7^2) \bmod 13 = (7^2 \bmod 13 * 7^2 \bmod 13) \bmod 13$$

$$7^4 \bmod 13 = (10 * 10) \bmod 13 = 100 \bmod 13 = 9$$

$$7^4 \bmod 13 = 9$$

$$7^8 \bmod 13 = (7^4 * 7^4) \bmod 13 = (7^4 \bmod 13 * 7^4 \bmod 13) \bmod 13$$

$$\vdots$$

# Modular exponentiation by squaring

$$\vdots$$

$$7^{256} \bmod 13 = (7^{128} * 7^{128}) \bmod 13 = (7^{128} \bmod 13 * 7^{128} \bmod 13) \bmod 13$$

$$7^{256} \bmod 13 = (3 * 3) \bmod 13 = 9 \bmod 13 = 9$$

$$7^{256} \bmod 13 = 9$$

# Modular exponentiation by squaring

Solve:

$$5^{117} \bmod 19$$

Check the solution at <https://tinyurl.com/gvq9vxz...>



# Outline

# Inverse modulo

What is a modular inverse?

- ▶ The modular inverse of  $A \pmod{C}$  is  $A^{-1}$ .
- ▶  $(A * A^{-1}) \equiv 1 \pmod{C}$  or equivalently  $(A * A^{-1}) \bmod C = 1$ .
- ▶ Only the numbers coprime (or relatively prime) to  $C$  (numbers that share no prime factors with  $C$ ) have a modular inverse  $\pmod{C}$ .

# Inverse modulo

Find the inverse for:

- ▶  $3 \pmod{7}$ .
- ▶  $2 \pmod{6}$ .

Solution at <https://tinyurl.com/hgxskmk...>

# Linear congruence modulo

Solve:

►  $17x \equiv 1 \pmod{43}.$

# Linear congruence modulo

$$17x \equiv 1 \pmod{43}.$$

$$17^{-1} \cdot 17x \equiv 17^{-1} \pmod{43}.$$

$$x \equiv 17^{-1} \pmod{43}.$$

Now we find  $17^{-1} \pmod{43}$

$$a \cdot b \equiv 1 \pmod{m}$$

$$a \cdot 17 \equiv 1 \pmod{43}$$

$$a \cdot 17 = 43 \cdot b + 1$$

## Linear congruence modulo<sup>3</sup>

Now we find  $17^{-1} \pmod{43}$

$$a \cdot 17 \equiv 1 \pmod{43}$$

$$a \cdot 17 = 43 \cdot b + 1$$

$$a = 38, b = 15$$

So we have

$$38 \cdot 17 = 43 \cdot 15 + 1$$

$$38 \cdot 17 \equiv 1 \pmod{43}$$

and

$$17^{-1} = 38 \pmod{43}$$

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<sup>3</sup>Have a look at <https://tinyurl.com/y7jtxze4> for an alternative method...

# Linear congruence modulo

$$17x \equiv 1 \pmod{43}.$$

$$17^{-1} \cdot 17x \equiv 17^{-1} \pmod{43}.$$

$$x \equiv 17^{-1} \pmod{43}.$$

We know that:  $17^{-1} = 38 \pmod{43}$ , so:

$$x \equiv 38 \pmod{43}$$

# Outline



# Fermat's Little Theorem

Let's have a look at <https://tinyurl.com/14ta3ym>

# Fermat's Little Theorem

If  $p \nmid a$ , then  $a^{p-1} \equiv 1 \pmod{p}$ .

Example:  $a = 5, p = 23$ .

So,  $5^{22} \equiv 1 \pmod{23}$ .

# Fermat's Little Theorem

$$5^{22} \equiv 1 \pmod{23}.$$

We compute this large power of 5 by successive squaring...

$$5^2 \equiv 25 \equiv 2 \pmod{23}$$

$$5^4 \equiv 2^2 \equiv 4 \pmod{23}$$

$$5^8 \equiv 4^2 \equiv 16 \pmod{23}$$

$$5^{16} \equiv 16^2 \equiv 256 \equiv 26 \equiv 3 \pmod{23}$$

Hence we have...

$$5^{22} \equiv 5^{16+4+2} \equiv 5^{16} \cdot 5^4 \cdot 5^2 \equiv 3 \cdot 4 \cdot 2 \equiv 24 \equiv 1 \pmod{23}$$

# Webography

1. Khan Academy - Journey into Cryptography  
Modular arithmetic  
<https://tinyurl.com/jvqfq8t>
2. Khan Academy - Journey into Cryptography  
Randomized algorithms  
<https://tinyurl.com/l4ta3ym>

