CS/MATH 111, Discrete Structures - Fall 2018. Discussion 02 - Proof by Induction, Logarithms, Asymptotic Notation and Execution Time.

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Outline

Example 1

Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

1. **Basis step:** For $n = 0$, $2^0 = 1 = 2^1 - 1$ is true!
2. **Assumption step:** Let $n = k$, so

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

   holds...
3. **Inductive step:** Let's solve for $n = k + 1$,

$$1 + 2 + 2^2 + \cdots + 2^{k+1} = 2^{(k+1)+1} - 1$$

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Example 2

Prove the following statement by induction:

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

1. **Basis step:** For $n = 1$, $1 = \frac{1 \times 2 \times 3}{6}$ is true!
2. **Assumption step:** Let $n = k$, so

$$1 + 2^2 + 3^2 + \cdots + k^2 = \frac{k \cdot (k+1) \cdot (2k+1)}{6}$$

holds...

3. **Inductive step:** Let's solve for $n = k + 1$,

$$1 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1) \cdot ((k+1)+1) \cdot (2 \cdot (k+1)+1)}{6}$$

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$$(k+1)\cdot(2k^2 + 7k + 6) \stackrel{?}{=} (k+1)\cdot(k+2)\cdot(2k+3)$$

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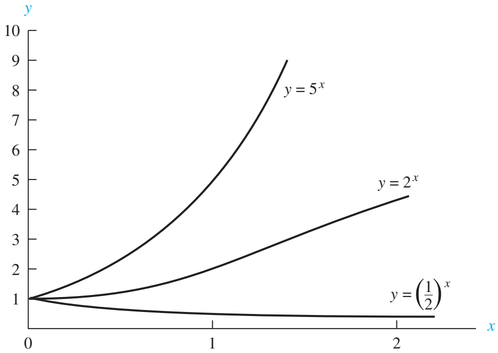
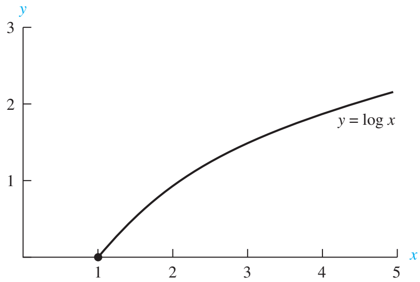
Outline

Exponential functions



Logarithmic functions



Theorems

Theorem 1

Let b be a positive real number and x and y real numbers. Then,

1. $b^{x+y} = b^x \cdot b^y$, *and*
2. $(b^x)^y = b^{x \cdot y}$.

Theorems

Theorem 2

Let b be a real number greater than 1. Then,

1. $\log_b(xy) = \log_b x + \log_b y$ *whenever x and y are positive real numbers, and*
2. $\log_b(x^y) = y \log_b x$ *whenever x is a positive real number and y is a real number.*

Theorems

Theorem 3

Let a and b be real numbers greater than 1, and let x be a positive real number. Then,

1. $\log_a x = \frac{\log_b x}{\log_b a}$.

Outline

Big-$\mathcal{O}$ notation

Definition 3.1

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\mathcal{O}(g(x))$ if there are constants C and k such that,

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-oh of $g(x)$."]

Big-Ω notation

Definition 3.2

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that,

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-omega of $g(x)$."]

Big-Θ notation

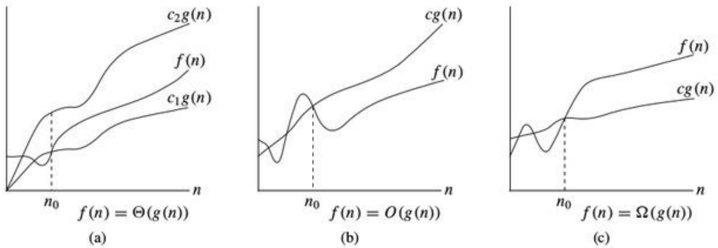
Definition 3.3

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $\mathcal{O}(g(x))$ and $f(x)$ is $\Omega(g(x))$.
Also note that $f(x)$ is $\Theta(g(x))$ iif there are real numbers C_1 and C_2 and a positive real number k such that,

$$C_1|g(x)| \leq |f(x)| \leq C_2|g(x)|$$

whenever $x > k$. [This is read as "$f(x)$ is big-theta of $g(x)$."]

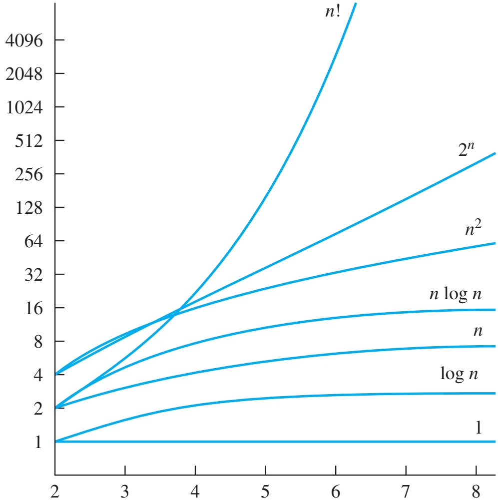
Asymptotic notation [1]



[1] Great additional resources at `https://tinyurl.com/o5lwvgp`

Outline

Growth of functions



Complexity of algorithms

TABLE 1 Commonly Used Terminology for the Complexity of Algorithms.

Complexity	*Terminology*
$\Theta(1)$	Constant complexity
$\Theta(\log n)$	Logarithmic complexity
$\Theta(n)$	Linear complexity
$\Theta(n \log n)$	Linearithmic complexity
$\Theta(n^b)$	Polynomial complexity
$\Theta(b^n)$, where $b > 1$	Exponential complexity
$\Theta(n!)$	Factorial complexity

Example 1

- Give a big-$\mathcal{O}$ estimate for the number of operations (where an operation is an addition or a multiplication) used in this segment of an algorithm.

$$
\begin{array}{l}
t := 0 \\
\textbf{for } i := 1 \textbf{ to } 3 \\
\quad \textbf{for } j := 1 \textbf{ to } 4 \\
\quad\quad t := t + ij
\end{array}
$$

Example 1

- The statement **t := t + ij** is executed just 12 times, so the number of operations is $\mathcal{O}(1)$. (Specifically, thereare just 24 additions or multiplications.)

$$
\begin{array}{l}
t := 0 \\
\textbf{for } i := 1 \textbf{ to } 3 \\
\quad \textbf{for } j := 1 \textbf{ to } 4 \\
\qquad t := t + ij
\end{array}
$$

Example 2

- Give a big-$\mathcal{O}$ estimate the number of operations, where an operation is a comparison or a multiplication, used in this segment of an algorithm (ignoring comparisons used to test the conditions in the for loops, where $a_1, a_2, \ldots, a_n$ are positive real numbers).

$$
\begin{array}{l}
m := 0 \\
\textbf{for } i := 1 \textbf{ to } n \\
\quad \textbf{for } j := i + 1 \textbf{ to } n \\
\quad\quad m := \max(a_i a_j, m)
\end{array}
$$

Example 2

- The nesting of the loops implies that the assignment statement is executed roughly $\frac{n^2}{2}$ times. Therefore the number of operations is $\mathcal{O}(n^2)$.

$m := 0$
for $i := 1$ **to** n
  for $j := i + 1$ **to** n
    $m := \max(a_i a_j, m)$

Example 3

```
for i:=1 to n do
    for j:=i to n do
        write('OK');
```

- $\frac{n(n+1)}{2}$
- $\mathcal{O}(n^2)$

Reference

- Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.
  Appendix 2: Exponential and Logarithmic functions.
  Chapter 3: Algorithms.
  Section 3.2: The Growth of Functions.
  Section 3.3: Complexity of Algorithms.