Simple Strategies for Large Zero-Sum Games with Applications to Complexity Theory

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Abstract

Von Neumann's Min-Max Theorem guarantees that each player of a zero-sum matrix game has an optimal mixed strategy. We show that each player has a *near-optimal* mixed strategy that chooses *uniformly* from a multiset of pure strategies of size logarithmic in the number of pure strategies available to the opponent. Thus, for exponentially large games, for which even *representing* an optimal mixed strategy can require exponential space, there are near-optimal, linear-size strategies. These strategies are easy to play and serve as small witnesses to the approximate value of the game.

Because of the fundamental role of games, we expect this theorem to have many applications in complexity theory and cryptography. We use it to strengthen the connection established by Yao between randomized and distributional complexity and to obtain the following results: (1) Every language has *anti-checkers* — small hard multisets of inputs certifying that small circuits can't decide the language. (2) Circuits of a given size can generate random instances that are hard for all circuits of linearly smaller size. (3) Given an oracle M for any exponentially large game, the approximate value of the game and near-optimal strategies for it can be computed in $\Sigma_2^{P(M)}$. (4) For any NP-complete language L, the problems of (a) computing a hard distribution of instances of L and (b) estimating the circuit complexity of L are both in Σ_2^P.

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1 Introduction

Games play a fundamental role in many parts of theory. For example, cryptographic problems can often be viewed as games between those who wish to keep a secret and those who wish to discover it [8]. Many computational classes can be defined in natural ways as games: for example, PSPACE can be defined in this way [6]. Other times games arise in a slightly more subtle way. For example, questions about how hard it is to generate hard instances of some problem can be modeled as a game between the generator and the algorithm. Yao [18, 19] exploits this idea to prove lower bounds on randomized algorithms.

The classic result on games is the famous Min-Max Theorem of von Neumann [17], which guarantees that each player of a zero-sum game has an optimal mixed strategy. For exponentially large games, optimal strategies are generally exponentially large. In many cases, we need to know not only that an object exists but also that it is not too complex. Without this latter restriction we cannot use the object.

Simple strategies for large games. Our first result is a variant of von Neumann's Min-Max Theorem that shows that each player has a near-optimal mixed strategy that plays uniformly from a multiset of size logarithmic in the number of pure strategies available to the opponent. The proof is a surprisingly simple probabilistic argument similar to circuit derandomization techniques [1, 15]. However, the central nature of games in theory suggests that this simple result may have far-reaching consequences. This result was obtained independently by Althöfer [3].

Strengthening the connection between randomized and distributional complexities. This connection was first established by Yao [18, 19]. He considered a game where MIN's pure strategies are the deterministic algorithms in a given class, MAX's pure strategies are the inputs of a given size, and the payoff for a particular pair is the cost of the algorithm on the input. If MIN moves first, the expected payoff can be interpreted as the worst-case expected complexity of the best randomized algorithm. If MAX moves first, the expected payoff can be interpreted as the average-case complexity of the best algorithm for the hardest input distribution. By von Neumann's theorem, these are the same. Thus, the worst-case complexity of the best randomized algorithm equals the optimal average-case complexity against the hardest input distribution.

A main drawback is that for equality to hold, the "randomized algorithms" must generally be allowed to have exponentially large encodings. Because of this, Yao's theorem has been used mainly in the weaker direction: to prove lower bounds on randomized complexity and upper bounds on average-case complexity. The stronger direction (equality) holds only for complexity measures that allow program size to grow *exponentially* with input size.

Our variant of the Min-Max theorem reduces the dependence on encoding size. Our variant implies that it suffices to consider randomized algorithms that have *linear-size* encodings. Thus, the stronger direction holds (approximately) for complexity measures that allow program size to grow linearly with input size. This includes most measures of circuit complexity.

Note that this application is similar to known circuit derandomization techniques [1, 15]. However, the theorem has many other applications. For instance, by applying it to the program/input game for the *input* player we show that there are hard distributions that can be *generated* by small circuits.

Anti-checkers and circuits that generate hard random instances. We give applications concerning the complexity of generating and solving hard random instances of problems. Our main application is to show that every language has *anti-checkers* — small multisets of inputs such that correctly classifying a fraction of the inputs in the multiset is nearly as hard as correctly classifying *all* inputs of the given size. Circuits of a given size can use anti-checkers to generate random instances that are hard for all slightly smaller circuits.

Uniform complexity. We obtain related results for uniform complexity measures. Specifically, we show that the following problems are in Σ_2^P:

- estimating the value of any exponentially large game given an oracle for the payoffs;
- computing approximate upper and lower bounds on the circuit complexity of L and
- computing hard random instances of L,

where L is any NP-complete language.

2 Other Related Work

Theorem 2, our first variant of von Neumann's Min-Max Theorem, was obtained independently by Althöfer [3]. He considers applications to other linear programs, large game trees, and uniform sampling spaces.

A subsequent work [20] gives simple greedy algorithms that (given the payoff matrix) find the k-uniform strategies shown to exist in Theorems 2 and 3.

Uniform complexity. As mentioned previously, the complexity class PSPACE has a natural characterization via games. More recently, the complexity classes NEXP and coNEXP have been similarly characterized [7]. Our variant of von Neumann's Theorem can be used in these characterizations of NEXP and coNEXP.

Most research on hard distributions to date concerns uniform complexity. A significant body of work concerns average-case *completeness*, e.g, [9, 16, 10, 12, 4]. These results are analogous to NP-completeness results, except they concern distributional problems — ⟨problem, input distribution⟩ pairs. These results relate the complexities of classes of distributional problems. Generally, few relations to worst-case complexity are known (see, however, [4]).

Ben-David et al. [4] and Li and Vitanyi [13] show the existence of distributions under which the average-case complexity of any program is within a constant (exponential in the size of the program) of the worst-case complexity. Generating random instances from such distributions is difficult — it requires diagonalizing against all programs in question. The result applies to uniform complexity classes, not circuits. More precisely, it generates inputs that are hard only for programs that are exponentially smaller than the inputs.

Circuit complexity. Schapire [14] shows that his technique for boosting the correctness of PAC-learning strategies can also be applied to boost the correctness of circuits. This implies the existence of distributions from which random instances are nearly as hard for circuits as worst-case instances. His results establish a version of Corollary 8, weaker in that the complexity of *generating* the distribution is not known and in that the factor in bound (2) is a larger polynomial.

Upper bounds. One example of the use of the Min-Max Theorem in the stronger direction (to upper bound randomized complexity, measured, in this case, by the competitive ratio) is given by Alon, Karp, Peleg and West [2]. They show the existence of randomized k-server strategies by considering a certain zero-sum matrix game. The competitiveness of the strategy is related to the value of the game, which in turn depends on the underlying metric space.

3 Simple Strategies

A *two-player zero-sum game* is specified by an $r \times c$ matrix M and is played as follows. MIN, the row player, chooses a probability distribution p over the rows. MAX, the column player, chooses a probability distribution q over the columns. A row i and a column j are drawn randomly from p and q, and MIN pays M_{ij} to MAX. MIN plays to minimize the expected payment; MAX plays to maximize it. The rows and columns are called the *pure strategies* available to MIN and MAX, respectively, while the possible choices of p and q are called *mixed strategies*. The Min-Max Theorem states that playing first and revealing one's mixed strategy is not a disadvantage:

Theorem 1 ([17])

$$\min_p \max_j \sum_i p(i) M_{ij} = \max_q \min_i \sum_j q(j) M_{ij}$$

Note that the second player need not play a mixed strategy — once the first player's strategy is fixed, the expected payoff is optimized for the second player by some pure strategy. The expected payoff when both players play optimally is called the *value* of the game. We denote it $\mathcal{V}(M)$.

3.1 Simple strategies for large games.

Games that model computations are often exponentially large. Generally, the optimal strategies are the primal and dual solutions, respectively, to an $O(NM)$-size linear program. For exponentially large games, optimal strategies are generally too large to even represent. This motivates considering smaller mixed strategies:

Definition 1 *A mixed strategy is* k-uniform *if it chooses uniformly from a multiset of k pure strategies.*

We show that for k proportional to the *logarithm* of the number of pure strategies available to the opponent, each player has a near-optimal k-uniform strategy.

Let $M_{\min}$ and $M_{\max}$ denote $\min_{ij} M_{ij}$ and $\max_{ij} M_{ij}$, respectively. Recall that M is an $r \times c$ matrix.

Theorem 2 *For any $\epsilon > 0$ and $k \geq \ln(c)/2\epsilon^2$,*

$$\min_{p \in \mathcal{P}_k} \max_j \sum_i p(i) M_{ij} \leq \mathcal{V}(M) + \epsilon(M_{\max} - M_{\min}),$$

where $\mathcal{P}_k$ denotes the k-uniform strategies for MIN. *Equality holds only if $k = \ln(c)/2\epsilon^2$. The symmetric result holds for* MAX.

Proof Assume WLOG that $M_{\min} = 0$ and $M_{\max} = 1$. Fix $\epsilon > 0$ and $k > \ln(c)/2\epsilon^2$, and form S by drawing k times independently at random from MIN's optimal mixed strategy. For any fixed pure strategy j of the opponent, the probability that

$$\sum_{i \in S} \frac{1}{|S|} M_{ij} \geq \mathcal{V}(M) + \epsilon \tag{1}$$

is bounded by $e^{-2k\epsilon^2}$. This is because the left-hand side is the average of k independent random variables in $[0,1]$ with expected value at most $\mathcal{V}(M)$ [11].

By the choice of k, $e^{-2k\epsilon^2} < 1/c$. Thus, the expected number of the opponent's c pure strategies that satisfy (1) is less than 1. Since the number of such strategies is an integer, it must be zero for *some* S of size k. □

For many important games, $M_{\max} - M_{\min}$ is constant. In this case, the theorem says that for any ϵ, MIN has an $O(\log c)$-uniform strategy that is within ϵ of optimal.

To model *dovetailing* computations, we give the following variant, in which MIN plays a small subset of pure strategies (called a *dovetailing set*) simultaneously, choosing the best once MAX commits to a play.

Theorem 3 *For $\epsilon > 0$ and $k \geq \log_{1+\epsilon} c$,*

$$\min_{|S|=k} \max_j \min_{i \in S} M_{ij} \leq \mathcal{V}(M) + \epsilon(\mathcal{V}(M) - M_{\min}).$$

Equality holds only if $k = \log_{1+\epsilon} c$. The symmetric result holds for MAX.

We omit the proof, which is similar to the proof of Theorem 3.

4 Distributional vs. Randomized Complexity

We next consider Theorems 2 and 3 in the context of the program/input game introduced by Yao.

Definitions 2 *Fix a finite class $\mathcal{P}$ of programs, a finite class $\mathcal{I}$ of inputs and a function $M : \mathcal{P} \times \mathcal{I} \rightarrow \Re$ (where $M(i,j)$ represents some cost of the computation $i(j)$).*

The (unlimited) randomized complexity *of M is $\min_p \max_{j \in \mathcal{I}} \sum_i p(i) M(i,j)$, where p ranges over the probability distributions on $\mathcal{P}$.*

The (unlimited) distributional complexity *of M is $\max_q \min_{i \in \mathcal{P}} \sum_j q(j) M(i,j)$, where q ranges over the probability distributions on $\mathcal{I}$.*

The program/input *game for M is the two-player zero-sum game given by $M_{ij} = M(i,j)$ for $i \in \mathcal{P}$ and $j \in \mathcal{I}$.*

As Yao observed, von Neumann's theorem applied to the program/input game implies that the unlimited randomized complexity and the unlimited distributional complexity are equal to $\mathcal{V}(M)$. As a corollary of Theorem 2 applied for each player, we obtain the following.

Definitions 3 *A k-uniform randomized program is a randomized program obtained by playing uniformly from a multiset of k programs in $\mathcal{P}$.*

The k-uniform randomized complexity *of M is* $\min_p \max_{j \in \mathcal{I}} \sum_i p(i)M(i,j)$, *where p ranges over the k-uniform distributions on $\mathcal{P}$.*

The k-uniform distributional complexity *of M is* $\max_q \min_{i \in \mathcal{P}} \sum_j q(j)M(i,j)$, *where q ranges over the k-uniform distributions on $\mathcal{I}$.*

Corollary 4 *Let $\Delta = M_{\min} - M_{\max}$.*

1. *For any $\epsilon > 0$ and $k > \ln(|\mathcal{I}|) / 2\epsilon^2$, the k-uniform randomized complexity of M exceeds the unlimited randomized complexity by less than $\epsilon\Delta$.*

2. *For any $\epsilon > 0$ and $k > \ln(|\mathcal{P}|) / 2\epsilon^2$, the unlimited distributional complexity of M exceeds the k-uniform distributional complexity by less than $\epsilon\Delta$.*

A good k-uniform randomized program corresponds to a multiset of k programs such that, for any input, the average complexity of those programs on that input is close to the unlimited randomized complexity of M. A good k-uniform input distribution corresponds to a multiset of k inputs such that, for any program, the average complexity of that program on those inputs is close to the unlimited distributional complexity.

Sometimes it is also useful to consider small sets of programs such that, on any input, *some* program achieves a low complexity on that input. Similarly, one might want a small set of inputs such that any program has high complexity on *at least one* of the inputs in the set. We call such small sets *dovetailing sets.* As a corollary to Theorem 3, we obtain the following.

Corollary 5 1. *For any $\epsilon > 0$ and $k > \log_{1+\epsilon} |\mathcal{I}|$, there exists a set of at most k programs such that, for any input, the complexity of some program in the set is less than $\mathcal{V}(M) + \epsilon(\mathcal{V}(M) - M_{\min})$ on that input.*

2. *For any $\epsilon > 0$ and $k > \log_{1+\epsilon} |\mathcal{P}|$, there exists a set of at most k inputs such that, for any program, the complexity of the program is more than $\mathcal{V}(M) - \epsilon(M_{\max} - \mathcal{V}(M))$ on some input in the set.*

5 Anti-checkers against circuits.

An *anti-checker for L against circuits of size s* is a multiset of inputs such that any circuit of size s fails to correctly classify (w.r.t. L) a fraction of the inputs in the multiset. Anti-checkers are similar to *program checkers* [5] (which verify program correctness on a *per-input* basis) in that anti-checkers allow certification of the complexity of L on a per-circuit basis.

We apply Corollary 4 is to show that, provided s is slightly less than the circuit size required to decide L without error, there are anti-checkers for L of size $O(s)$. As a consequence, we obtain small circuits that generate hard random inputs.

Other flavors of anti-checkers for various complexity measures and with different notions of "anti-checking" are possible. To illustrate the issues, at the end of this section we give a variation in which the anti-checker is a small set of inputs such that any program of a given size has a high *running time* on *at least one* of the inputs in the set.

The first form of anti-checker is obtained by applying Corollary 4 to a program/input game where the programs are the circuits of size s and the inputs are the binary strings of size n. (More generally, we could take the programs to be those with encoding i ($0 \leq i < 2^s$) and the inputs to be those with encoding j ($0 \leq j < 2^n$). We require only that the program encoding scheme satisfy some basic compositional properties.) We take the complexity measure to be correctness, i.e., the payoff of the program/input game is zero if the program is correct on the input and one otherwise.

As described below in the proof of Theorem 6, a k-uniform randomized program with worst-case probability of error less than $1/2$ yields a deterministic program of size $O(ks)$ that is correct on all inputs. Thus, for circuits just slightly smaller than the smallest circuit deciding membership without error, there are hard input distributions on which no such circuit achieves a probability of error significantly less than $1/2$. Further, there are such hard input distributions which are k-uniform for small k. The underlying multiset yields the desired anti-checker.

Definition 4 *Define C_L, the* circuit complexity *of language L, to be the function such that $C_L(n)$ is the size (length of the encoding in binary) of the smallest circuit deciding membership in L of all n-bit binary strings.*

Theorem 6 *There exists a number N such that, for any language L and numbers $n > N$, $\epsilon > 0$, and $s \leq C_L(n)\epsilon^2 / 3n$, there exists a multiset of s/ϵ^2 length n binary strings such that every circuit of size s misclassifies at least a fraction $1/2 - \epsilon$ of the strings in the multiset.*

Proof Let $M(i,j)$ be 0 if the ith size s circuit correctly decides whether the jth n-bit binary string is in L and 1 otherwise. Let $\delta = 1/2 - \mathcal{V}(M)$. The two parts of Corollary 4 respectively imply:

i. There are $1 + n\ln(2) / 2\delta^2$ circuits of size s such that on any n-bit string, a majority of the circuits classifies the string correctly.

ii. Provided $\epsilon > \delta$, there are $s\ln(2) / 2(\epsilon - \delta)^2$ n-bit strings such that any size s circuit misclassifies at least a fraction $1/2 - \epsilon$ of the strings.

From (i), it follows that there is a circuit of size $ns\ln(2) / 2\delta^2 + s + O(n/\delta^2)$ that correctly classifies all n-bit strings. (The circuit returns the majority of what the $n/2\delta^2$ circuits return.) Thus, $ns\ln(2) / 2\delta^2 + s + O(n/\delta^2) \geq \mathcal{C}_L(n)$. By the choice of s, this implies $\delta/\epsilon \leq \sqrt{\ln(2) / 6} + O(1/n)$. This implies that, for large enough n, the number of strings in (ii) is at most s/ϵ^2. □

For instance, taking $\epsilon = 1/3$, $n > N$ and $s \leq \mathcal{C}_L(n)/27n$, there exists a multiset of $9s$ inputs such that any circuit of size s errs on one sixth of the inputs in the multiset. Intuitively, the problem of computing all 2^n inputs correctly is harder than the problem of computing a fraction of a fixed multiset of inputs correctly. Thus, it is surprising that such hard multisets exist.

Note also the contrapositive: to show that $\mathcal{C}_L(n) \leq 27ns$, it suffices to exhibit, for every multiset of $9s$ inputs, a size s circuit that errs on less than one sixth of the inputs in the multiset. Note that the tradeoff here is close to tight: for any such multiset, some circuit of size $O(sn)$ correctly classifies every input in the multiset.

Similar results are possible for other complexity measures (e.g., running time, space, circuit depth, etc.). There are three general considerations:

1. Instead of considering *expected* complexity (e.g., the expected running time of a program), one considers the *probability* that the complexity exceeds a given threshold. This yields a game with small $M_{\max} - \mathcal{V}(M)$, which allows small anti-checkers.

2. For some complexity measures, to build a deterministic program that has low complexity on all inputs, it suffices to find a small set of programs such that, on any input, *at least one* (as opposed to a majority) of the programs in the set has low complexity.

3. One might be interested in a weaker form of anti-checker, one such that any program has high complexity on *at least one* (as opposed to a fraction) of the inputs in the set.

The following example illustrates these three considerations.

Definitions 5 *Let $\mathcal{P}_L(n,t)$ denote the size of the smallest program that decides language L in time t for all n-bit inputs.*

Theorem 7 *Fix any language L and numbers n, t and $s < \mathcal{P}_L(n,t)$. Let $\mathcal{I}$ be the inputs of size n; let $\mathcal{P}$ be the programs of size s that correctly decide L on inputs of size n.*

There exists a set $S \subseteq \mathcal{I}$ of size

$$O\left(\frac{s}{\log \frac{\mathcal{P}_L(n,O(tn))}{ns}}\right)$$

such that each program in $\mathcal{P}$ requires more than time t on some input in S.

Proof For $i \in \mathcal{P}$ and $j \in \mathcal{I}$, let $M(i,j)$ be zero if program i runs in time t on input j and one otherwise. The value of the program/input game for M is the minimum probability of exceeding time t by any program on a random input from the hardest input distribution. Let this value be $1 - \delta$. By Corollary 5,

i. Taking $\epsilon = 1/(1-\delta) - 1$, for $k = O(n/\delta)$, there exists a set of k programs such that, for any input, the complexity of some program in the set is less than $1 = (1-\delta)(1+\epsilon)$ on that input.

ii. Taking $\epsilon = 1/\delta - 1$, for $k = O(s/\log(1/\delta))$, there exists a size k set of inputs such that, for any program, the complexity of the program is more than $0 = 1 - \delta(1+\epsilon)$ on some input in the set.

By (i), there exists a program of size $O(ns/\delta)$ that correctly classifies each size n input in time $O(nt)$. This program simply dovetails the k programs in the set and returns when the first program finishes. (At least one of the programs finishes in time t.) Thus, $\delta = O(sn/\mathcal{C}_L(n, O(nt)))$.

By (ii), there exists a set of $O(s/\log(1/\delta))$ inputs such that any program of size s takes time at least t on at least one of the inputs in the set. □

5.1 Generating hard random instances.

An easy corollary of the existence of small anti-checkers is that circuits of a given size (up to the circuit complexity of the language) can *generate* random inputs that are hard for all slightly smaller circuits to classify correctly.

Definition 6 *For any probability distribution D on $\{0,1\}^*$, define $\mathcal{C}_{L,D}$,* the circuit complexity of deciding $\langle L, D\rangle$ with error, *to be the function such that $\mathcal{C}_{L,D}(n,\epsilon)$ is the size of the smallest circuit deciding membership in L with probability of error at most ϵ when given a random input drawn from D restricted to strings of size n.*

Corollary 8 *There exists an N such that, for any language L, and any numbers $n > N$, $0 < \epsilon \leq 1/2$, and*

$s' \leq \mathcal{C}_L(n)$, some circuit of size s' computes a distribution D such that

$$\mathcal{C}_{L,D}(n, 1/2-\epsilon) \geq \Omega(s'\epsilon^2/n) \quad (2)$$
$$\mathcal{C}_{L,D}(n,0) \leq O(s'). \quad (3)$$

The proof is a straightforward application of Theorem 6 — the circuit computes the uniform distribution on the anti-checker against circuits of size $\Omega(s'\epsilon^2/n)$.

5.2 Finding near-optimal strategies non-deterministically.

Small simple strategies approximately determine the value of any game. Thus, for exponentially large games, under the right conditions, one can non-deterministically verify the approximate value.

Theorem 9 *Given $\epsilon > 0$, r, c, and an $r \times c$ game M in the form of a poly-time oracle computing M_{ij} from i and j, the problem of computing for each player a mixed strategy that guarantees a payoff within $\epsilon(M_{\max} - M_{\min})$ of $\mathcal{V}(M)$ is in $\Sigma_2^{P(M)}$, where $P(M)$ means polynomial in $r + c + \log(rc)/\epsilon$.*

Proof We describe a $\Sigma_2^{P(M)}$ computation that guesses and verifies the strategies for both players simultaneously, assuming WLOG that $M_{\max} = 1$ and $M_{\min} = 0$.

Non-deterministically guess an approximate value v for $\mathcal{V}(M)$. By Theorem 2, there is a strategy for MIN that chooses uniformly from a multiset S of k pure strategies, where $k = O(\log(c)/\epsilon^2)$, and that guarantees a payoff less than $\mathcal{V}(M) + \epsilon/2$. Guess S non-deterministically. Use a non-deterministic oracle query to verify that

$$(\forall j) \sum_{i \in S} \frac{1}{|S|} M_{ij} \leq v + \epsilon/2.$$

Similarly, guess and verify a mixed strategy for MAX that guarantees a payoff of at least $v - \epsilon/2$.

Because each strategy guarantees an expected payoff within ϵ of that guaranteed by the other, each expected payoff is within ϵ of optimal. □

5.3 Estimating circuit complexity and generating hard distributions in $\Sigma_2^P \cap \Pi_2^P$.

Using the close relationship established in Theorem 6 between circuit complexity and the value of the program/input game defined there, we can in some sense specialize the preceding theorem to obtain the following result, which can be interpreted as showing that the circuit complexity of any NP-complete language can be approximated within a linear factor in $\Sigma_2^P \cap \Pi_2^P$.

Theorem 10 *1. For any NP language L, there exists a decision procedure A in Σ_2^P such that $A(n,s)$ accepts if $\mathcal{C}_L(n) \geq 3ns$ but rejects if $\mathcal{C}_L(n) \leq s$.*

2. For any NP-complete language L, there exists a decision procedure B in Σ_2^P such that $B(n,s)$ accepts if $\mathcal{C}_L(n) \leq s$ but rejects if $\mathcal{C}_L(n) > s$.

Here the class P is those languages decidable in time polynomial in n and s.

Proof Let $L = \{x : (\exists y)\, \ell(x,y)\}$ be defined by the poly($|x|$)-time predicate ℓ.

The decision procedure A non-deterministically guesses an anti-checker and uses standard techniques to verify it. By Theorem 6, if $\mathcal{C}_L(n) \geq 3ns$, then there are $O(s)$ inputs $\{x_i : i = 1, .., O(s)\}$ such that any circuit of size s misclassifies at least one input x_i. On the other hand, if $\mathcal{C}_L(n) \leq s$ then clearly no such set exists. Thus, the following predicate is true if $\mathcal{C}_L(n) \geq 3ns$ but false if $\mathcal{C}_L(n) \leq s$:

$$\begin{aligned}&(\exists x_1, ..., x_{O(s)})\ (\forall C)\ (\exists i)\\ &(C(x_i) = 0 \wedge (\exists y_i)\ell(x_i, y_i) = 1)\\ &\vee(C(x_i) = 0 \wedge (\forall z_i)\, \ell(x_i, z_i) = 0),\end{aligned}$$

where C ranges over all size s circuits and the x_i's range over the n-bit strings. The "$(\exists i)$" quantifies over polynomially many i, so it can be expanded into an appropriate poly-size formula. Using standard quantifier-elimination techniques, the resulting expression can be converted to the form $(\exists X, Y)(\forall C, Z)\ell'(n, s, C, X, Y, Z)$, where ℓ' is a poly(n, s)-time predicate. Thus, the predicate is in Σ_2^P.

We construct the decision procedure $B(n,s)$ using standard techniques. It is known that, since L is NP-complete, the circuit complexity of the "witness" function w such that $\ell(x, w(x))$ for $x \in L$ is only polynomially larger than $\mathcal{C}_L(n)$. Thus the following predicate holds iff $\mathcal{C}_L(n) \leq s$.

$$\begin{aligned}&(\exists C, W)(\forall x)\\ &(C(x) = 0 \wedge (\forall y)\ell(x,y) = 0)\\ &\vee(C(x) = 1 \wedge \ell(x, W(x)) = 1),\end{aligned}$$

where C ranges over all circuits of size s, W ranges over all circuits of size large enough to compute the witness function, and x ranges over all inputs of size n. This predicate is clearly in Σ_2^P. □

5.4 Hard distributions for uniform complexity classes.

The first part of the proof of Theorem 10 can easily be modified to show that hard distributions for NP-complete languages can be computed in Σ_2^P. This gives the following result.

Proposition 11 *Assume the polynomial-time hierarchy doesn't collapse to Σ_2^P and let $k > 0$. For any NP- or co-NP-complete language L there is a distribution D on n-bit strings such that*

- *D is computable in Σ_2^P*
- *no $O(n^k)$-time algorithm (even with $O(n^k)$ advice), when given a random input from D, decides membership in L with probability of error less than $1/2 - 1/n^k$.*

We leave the proof to the full paper.

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