On a Linear Program for Minimum Weight Triangulation

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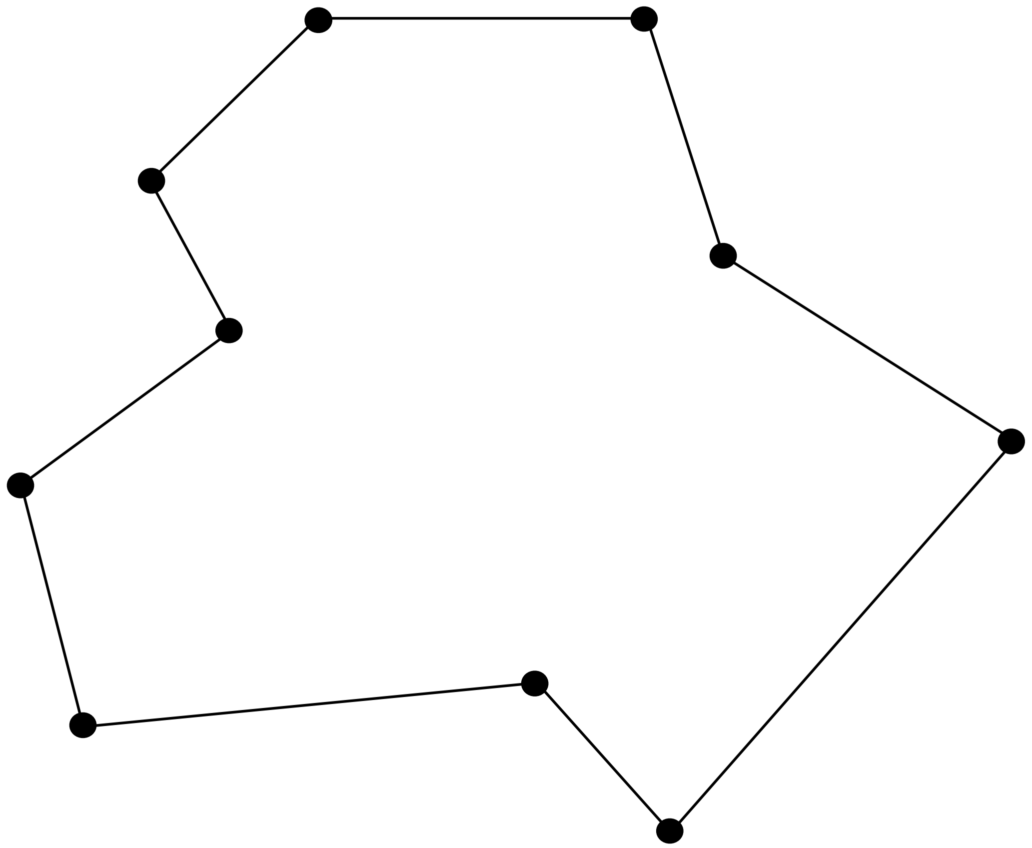
University of California, Riverside

full paper @ SODA 2012 / arxiv.org

min-weight triangulation of a **simple polygon**



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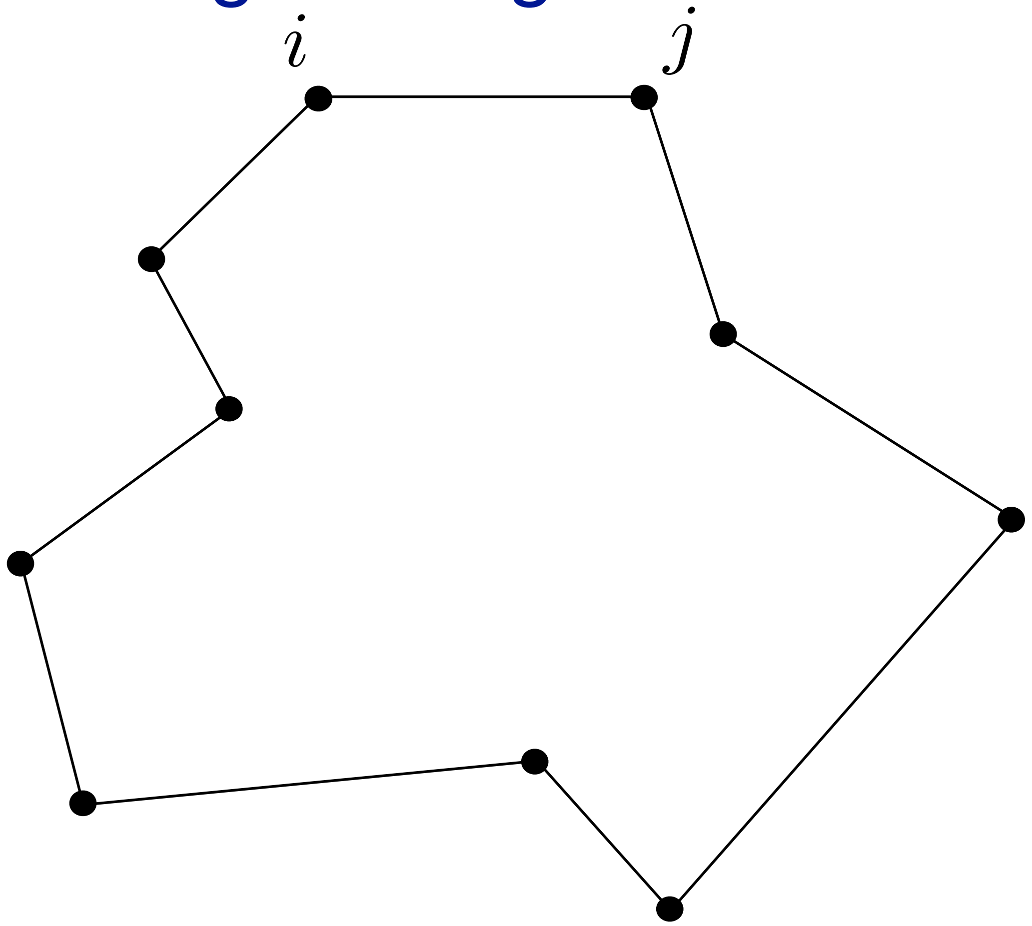

- dynamic programming
- $O(n^3)$ time

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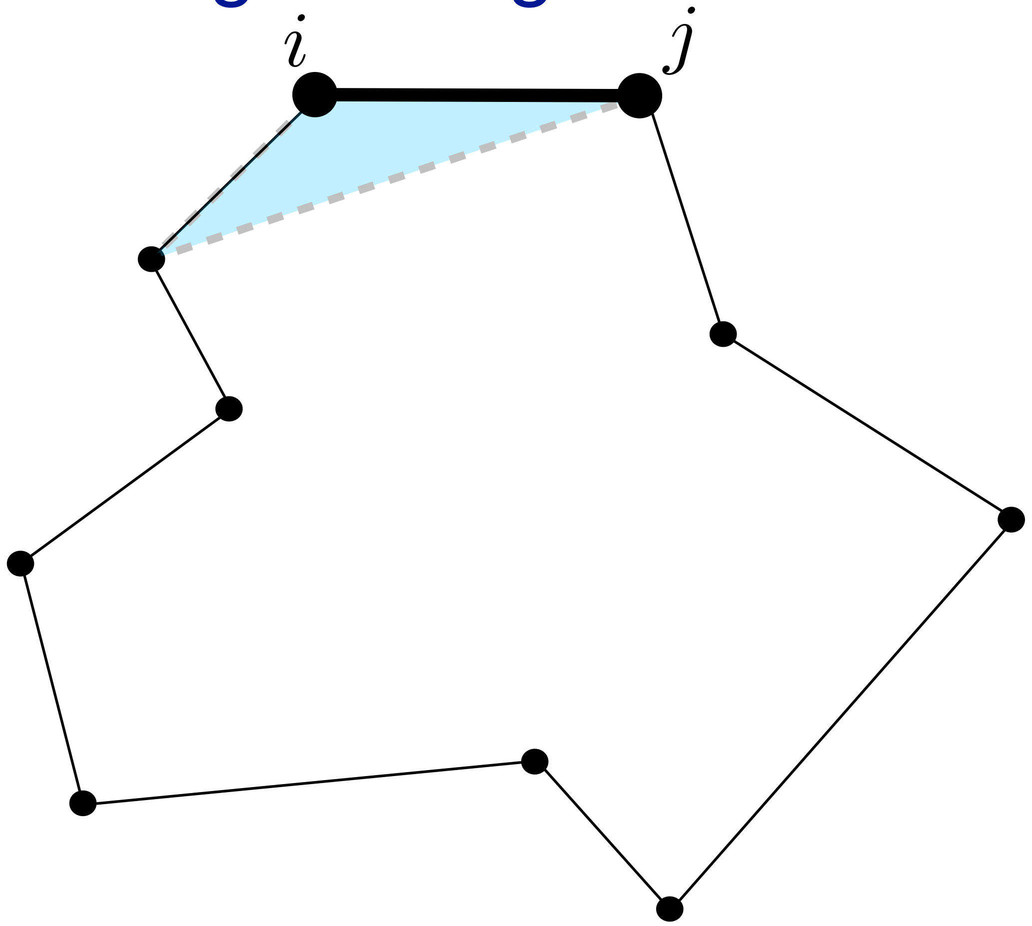

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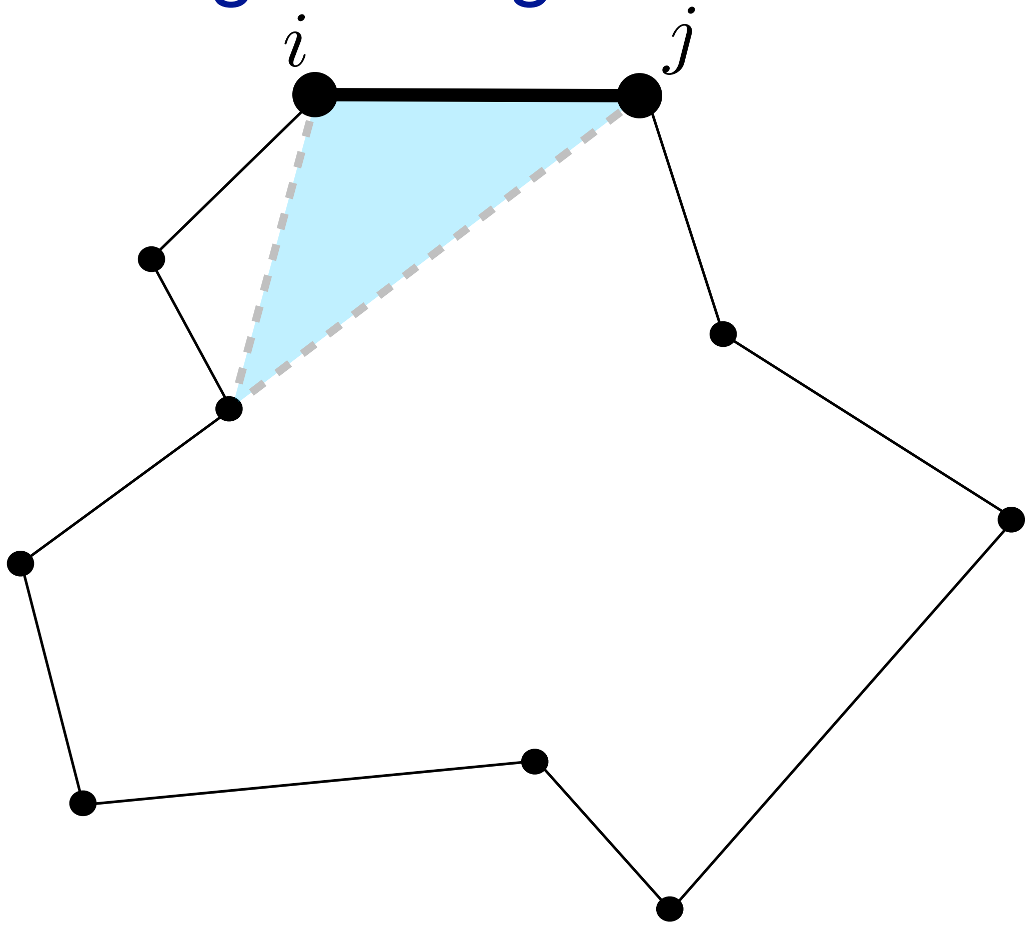

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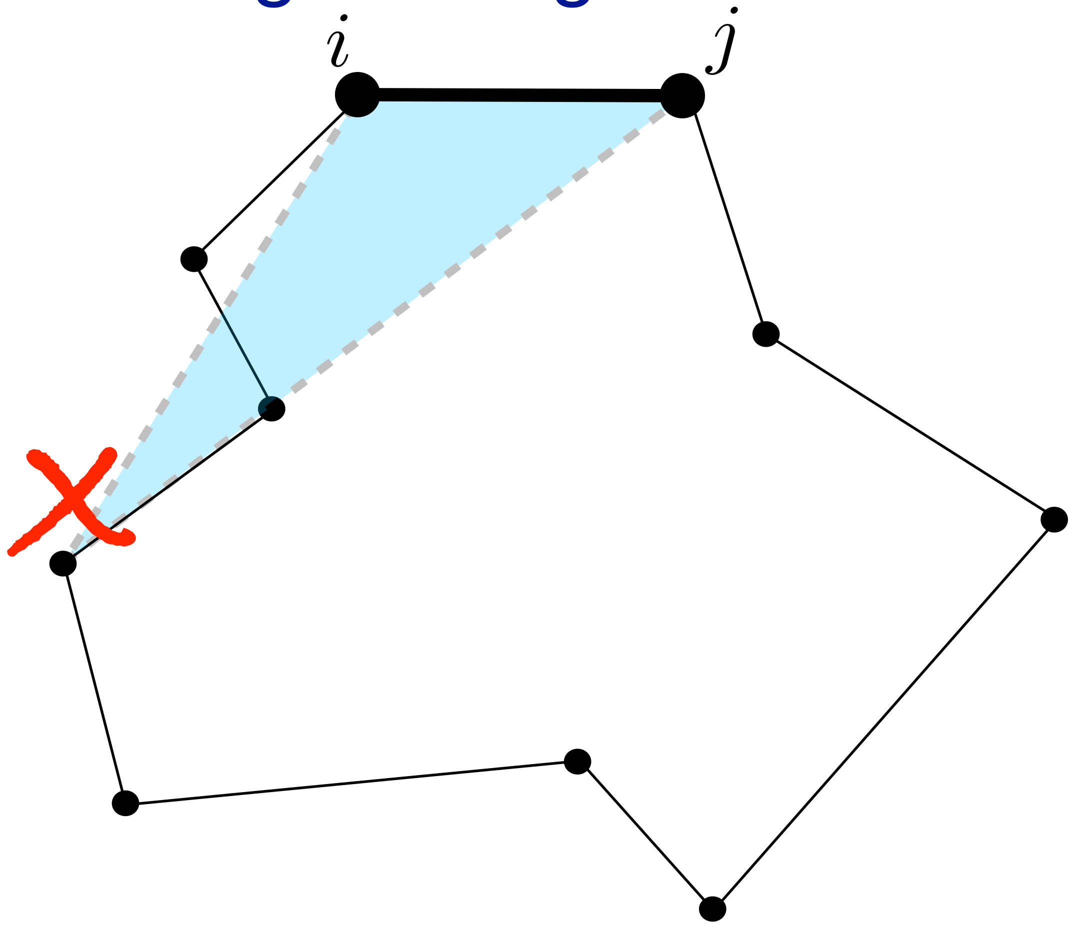

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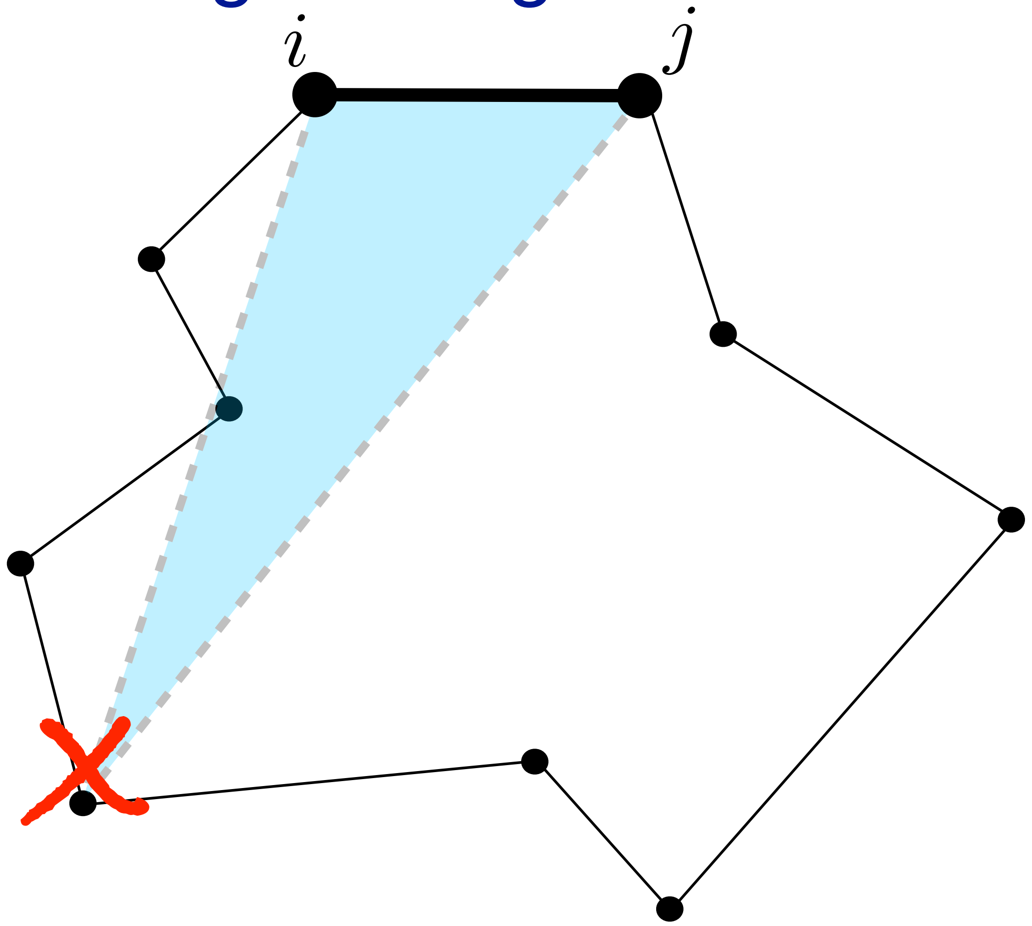

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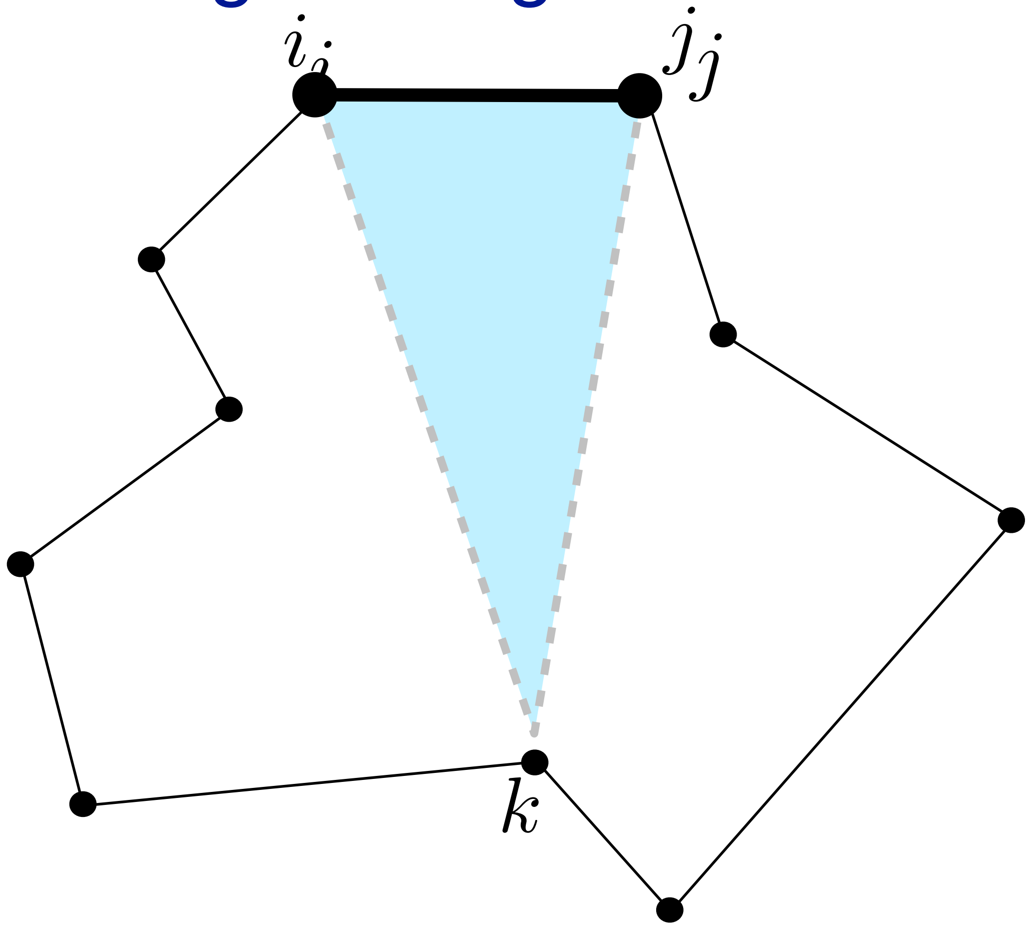
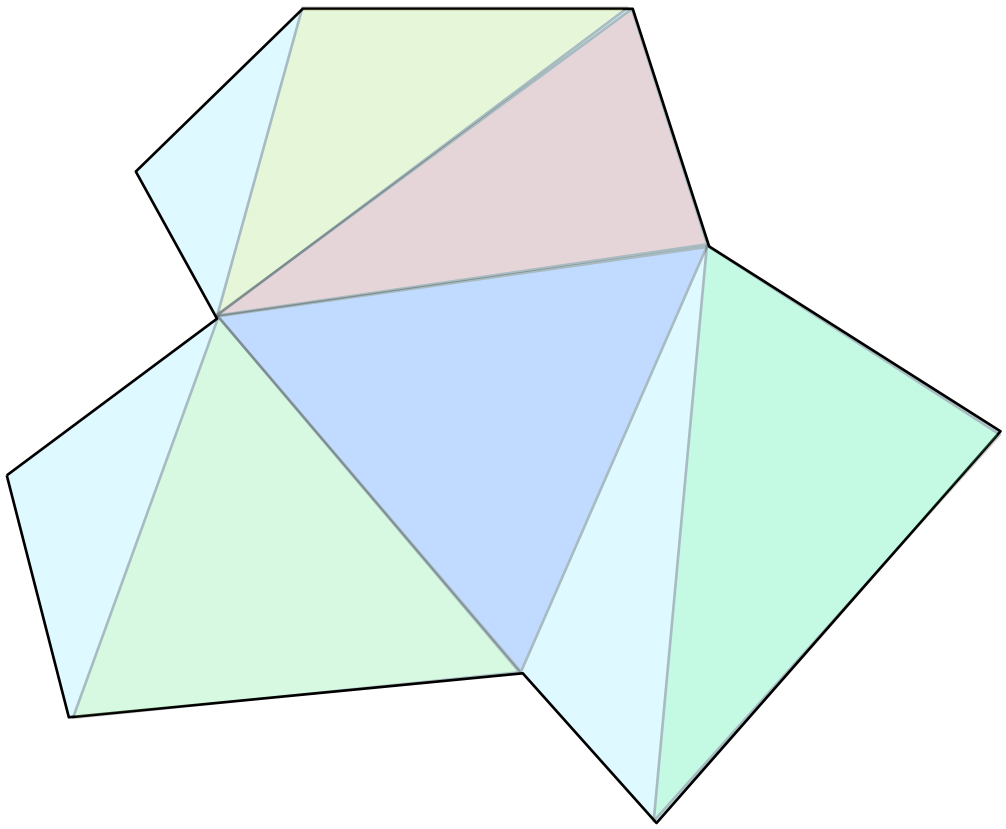


- dynamic programming
- $O(n^3)$ time

$$M[i,j] = \min\Big\{ M[i,k] + M[k,j] + d(i,j) \;\Big|\; i < k < j \Big\}$$

$$M[i,i+1] = d(i,i+1)$$

min-weight triangulation of a **simple polygon**

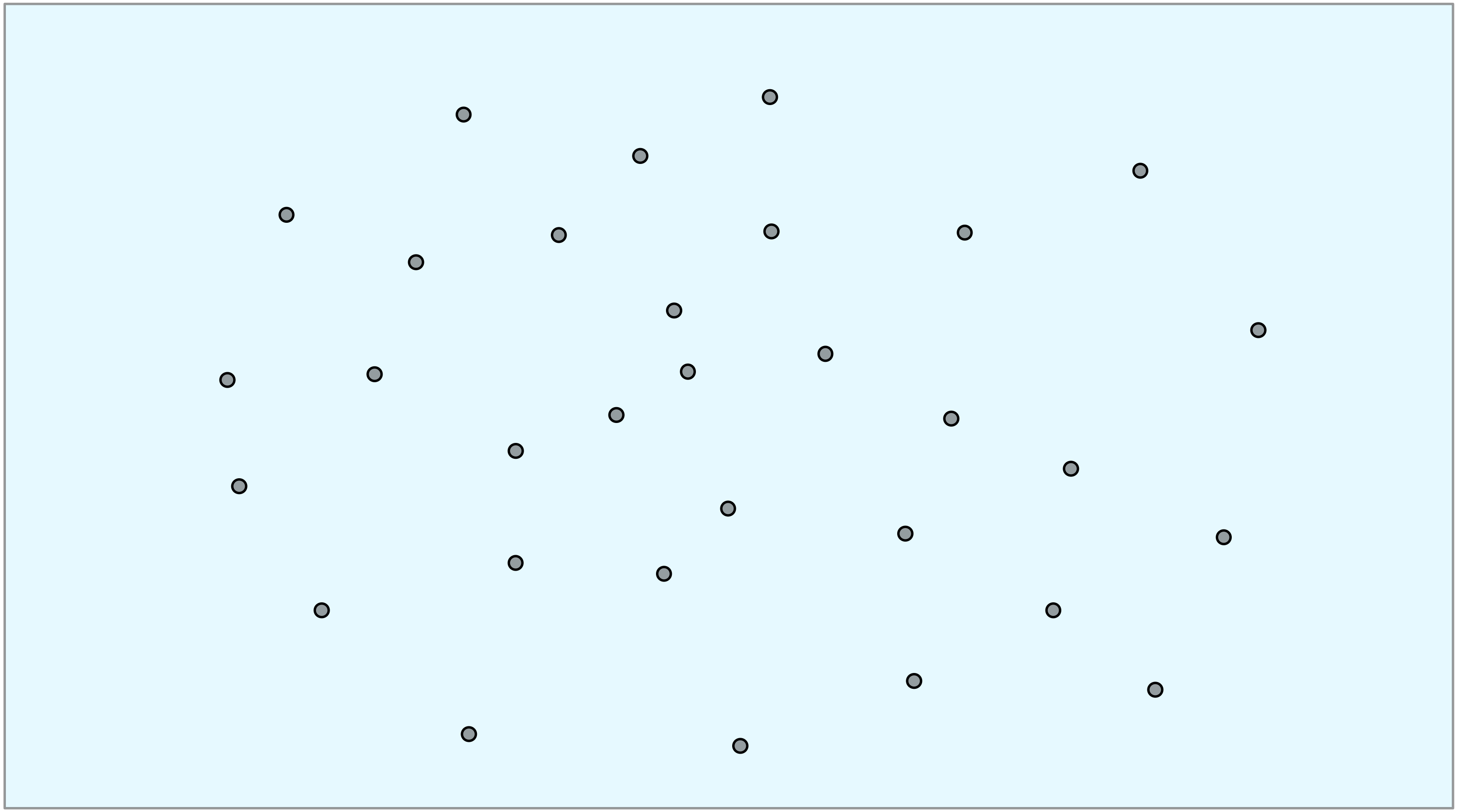


- dynamic programming
- $O(n^3)$ time

- [1979] Gilbert. *New results on planar triangulations.*
- [1980] Klincsek. *Minimal triangulations of polygonal domains.*

minimum weight triangulation (MWT)

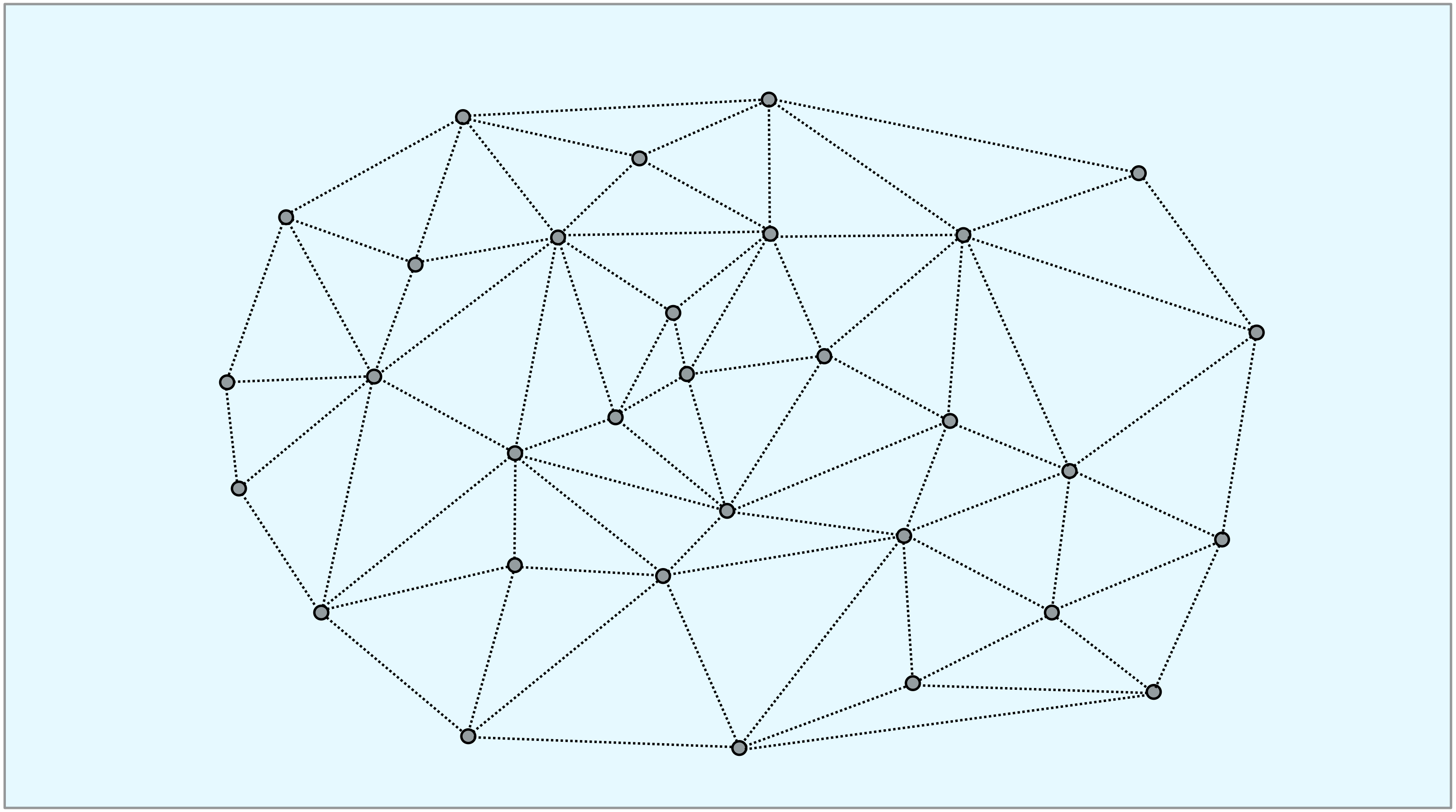
input: a set of points in the plane:



output:

minimum weight triangulation (MWT)

input: a set of points in the plane:



output: a triangulation T

minimum weight triangulation (MWT)

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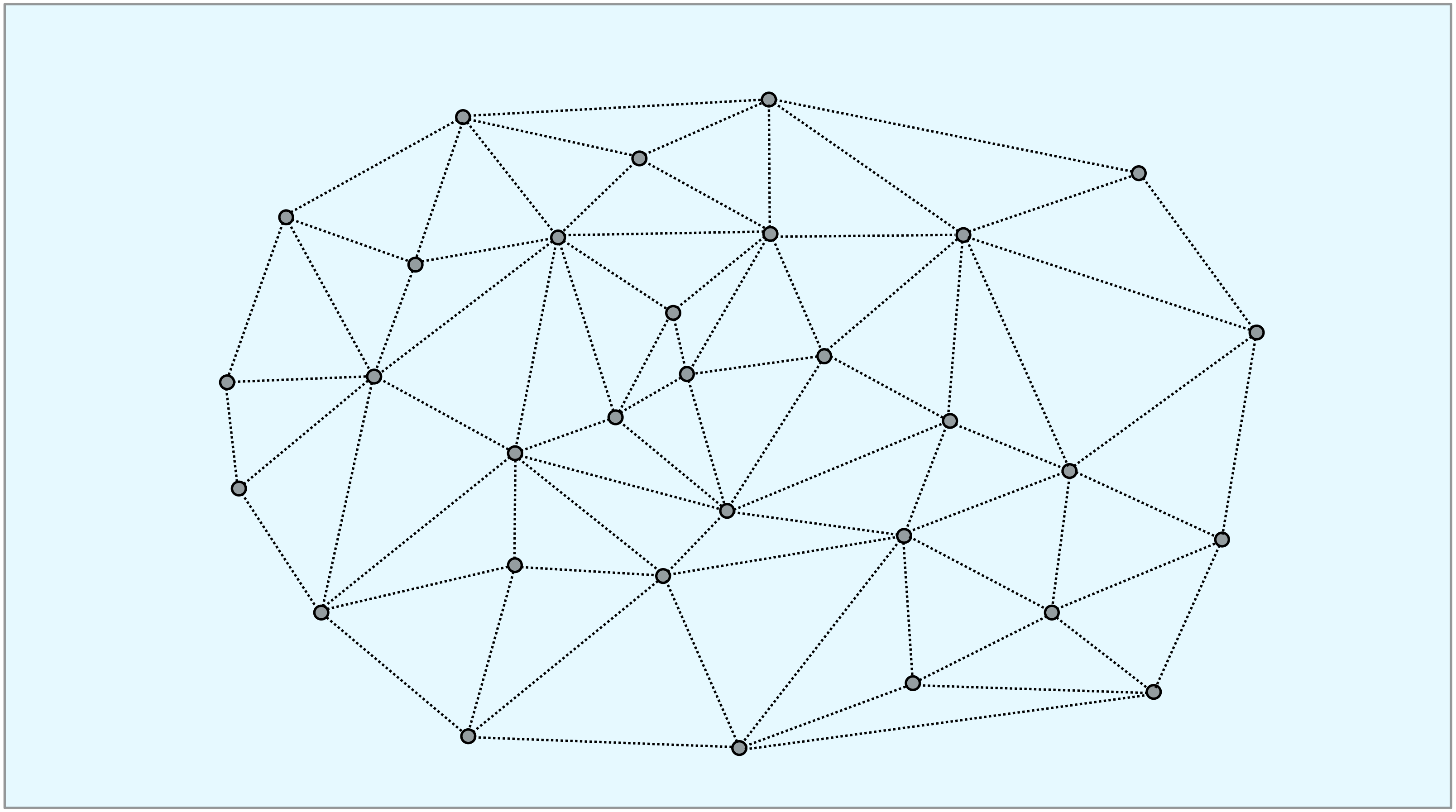

output: a triangulation T of minimum weight, $\sum_{e \in T} |e|$

THE BIBLE (1979)

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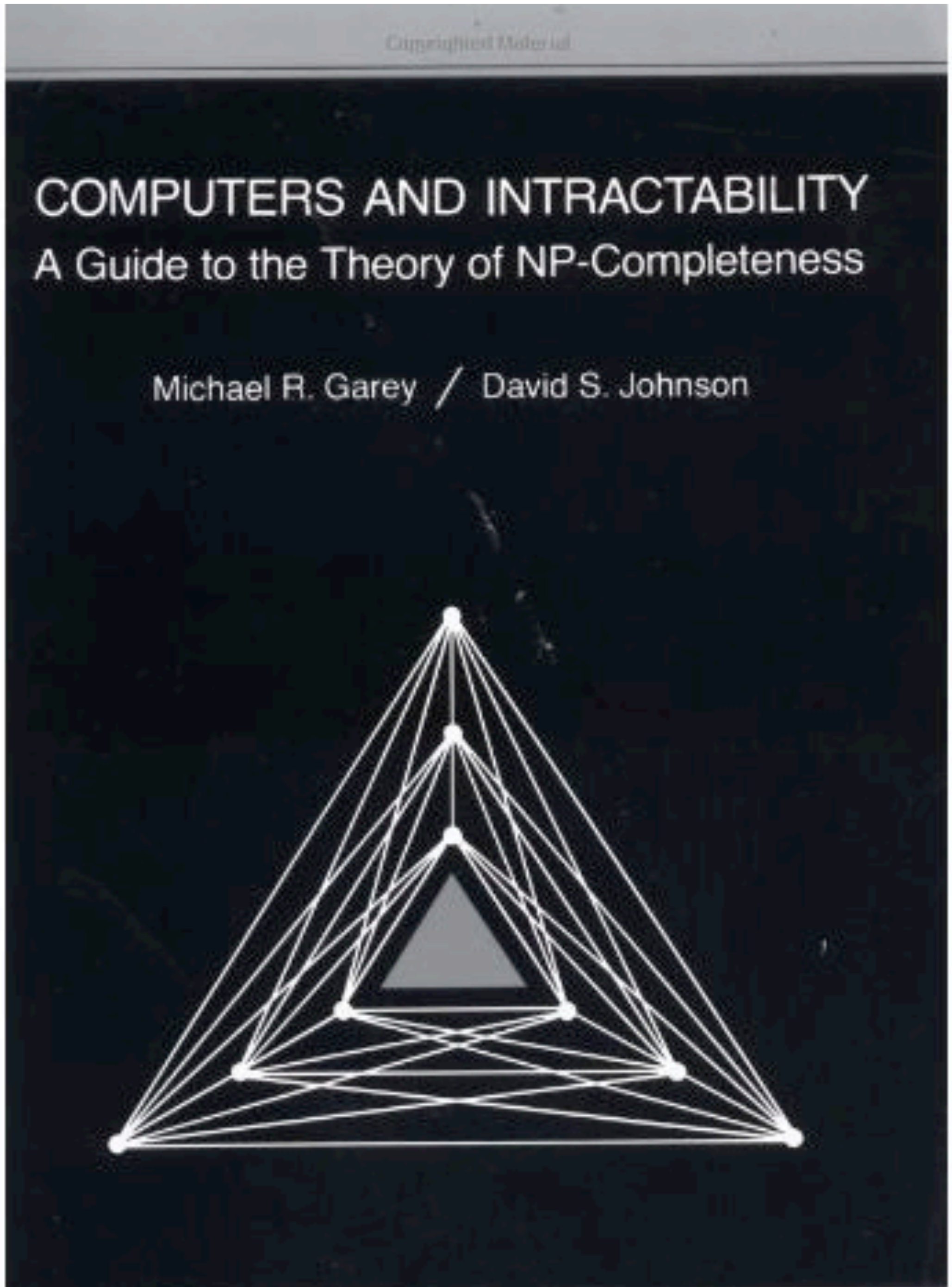


THE BIBLE (1979)

Contents

THE BIBLE (1979)

MWT
NP-Hard?
In P?

approximation algorithms

- [1987] Plaisted and Hong. O(log n)-approx
  A heuristic triangulation algorithm.

- [1996] Levcopoulos and Krznaric. O(1)-approx
  Quasi-greedy triangulations approximating the minimum weight triangulation.

- [2006] Remy and Steger. QPTAS
  A quasi-polynomial time approximation scheme for minimum weight triangulation.

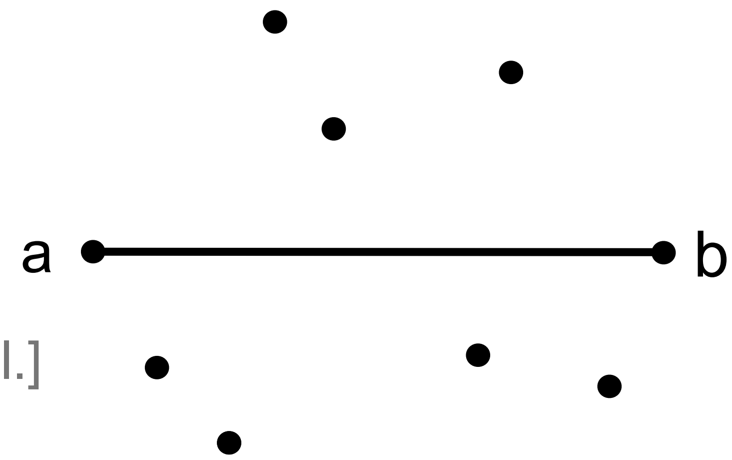
hardness result

- [2006] Mulzer and Rote. 25 years after G+J! NP-HARD
  Minimum weight triangulation is NP-hard.

heuristics!

- edges that can't be in any MWT:
  - diamond test

    [1989 Das and Joseph; 2001 Drysdale et al.]



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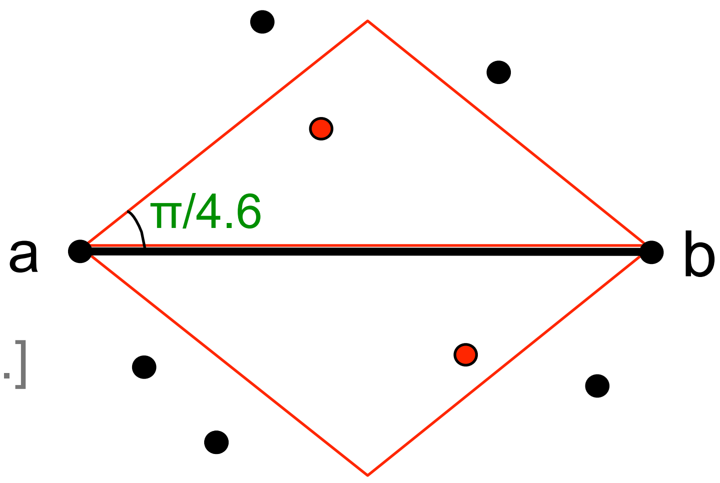

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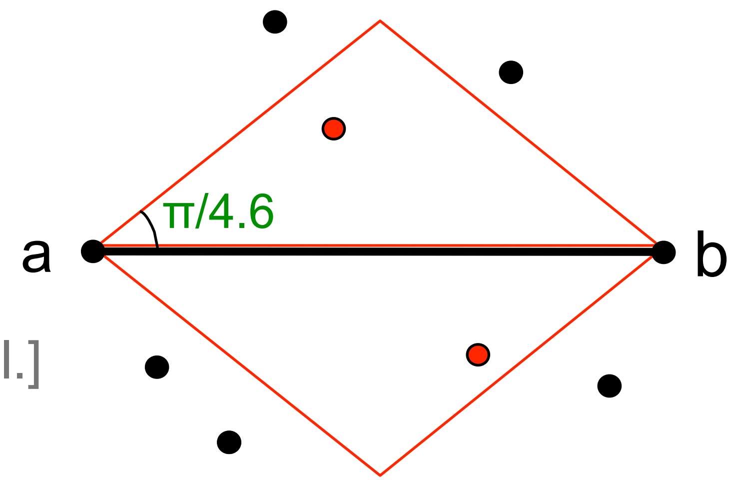
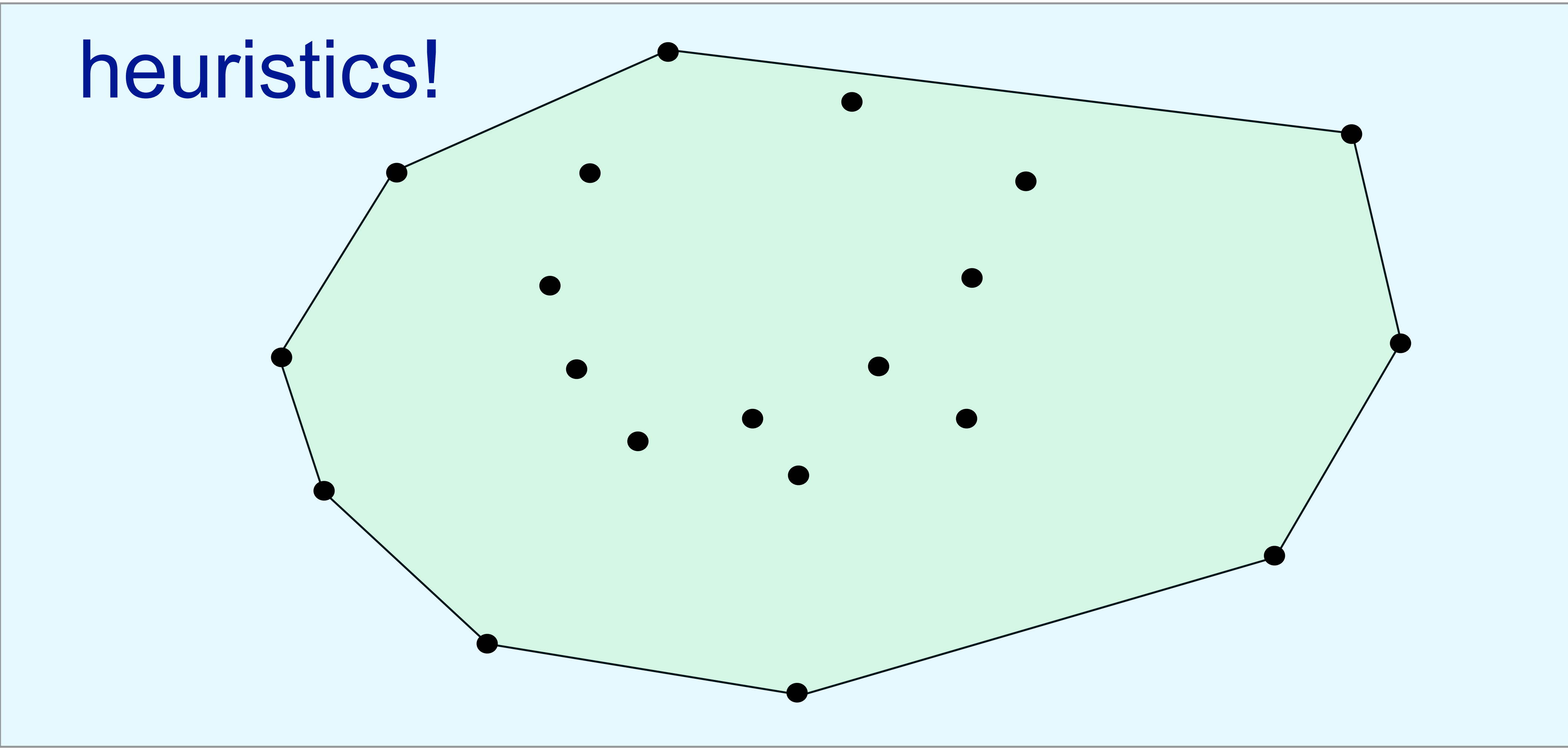


- edges that *can't* be in any MWT:
  - diamond test

    [1989 Das and Joseph; 2001 Drysdale et al.]

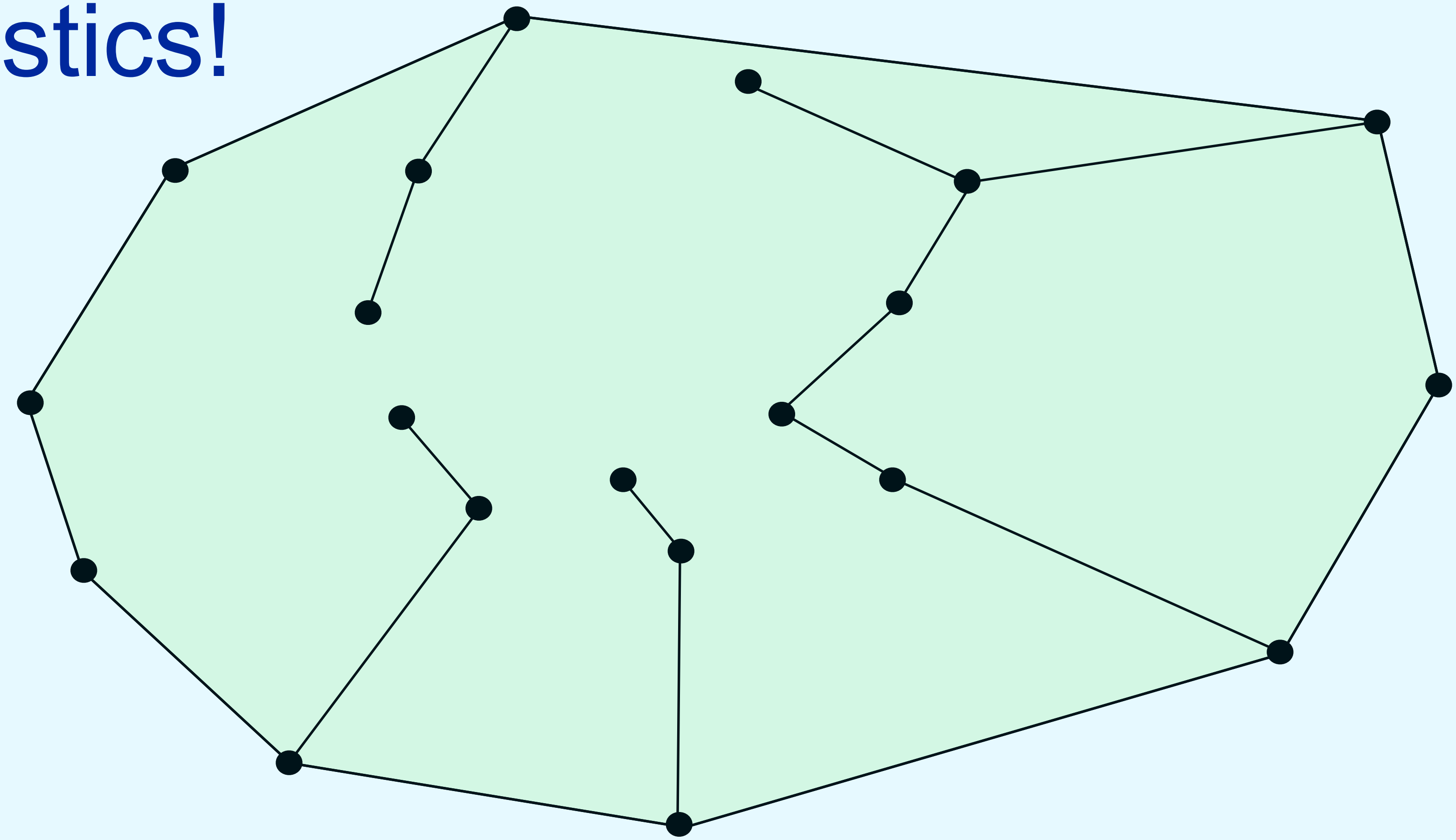
- edges that *have to be* in every MWT:
  - mutual nearest neighbors [1979 Gilbert; 1994 Yang et al]
  - β-skeleton [1993 Keil; 1995 Yang; 1996 Cheng and Xu]
  - locally minimal triangulation (“LMT-skeleton”)

    [1997 Dickerson et al; 1998 Beirouti and Snoeyink; 1996 Cheng et al; 1999 Aichholzer et al; 1996 Belleville et al; 2002 Bose et al]



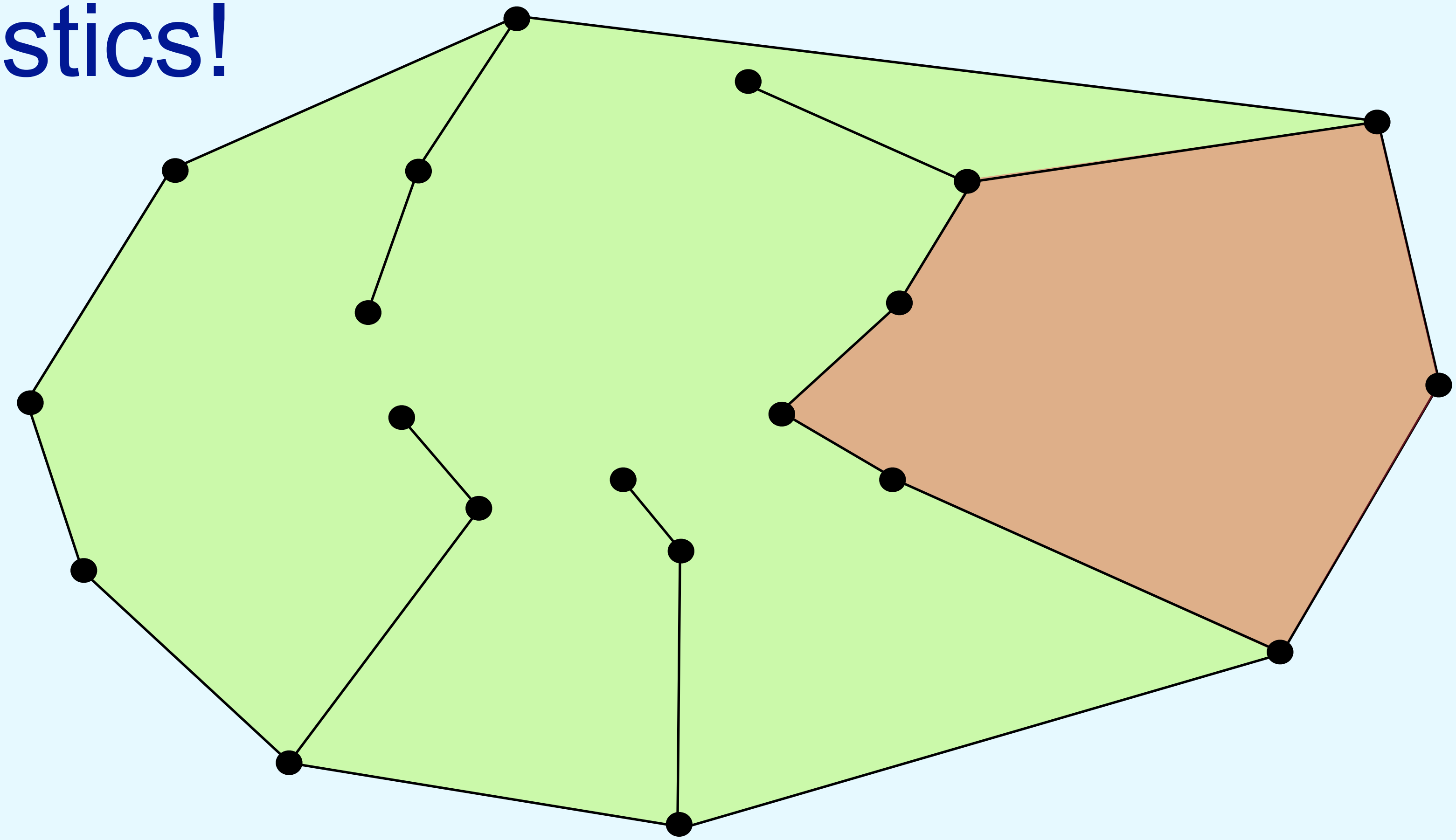
1. The boundary edges have to be in the MWT.

heuristics!

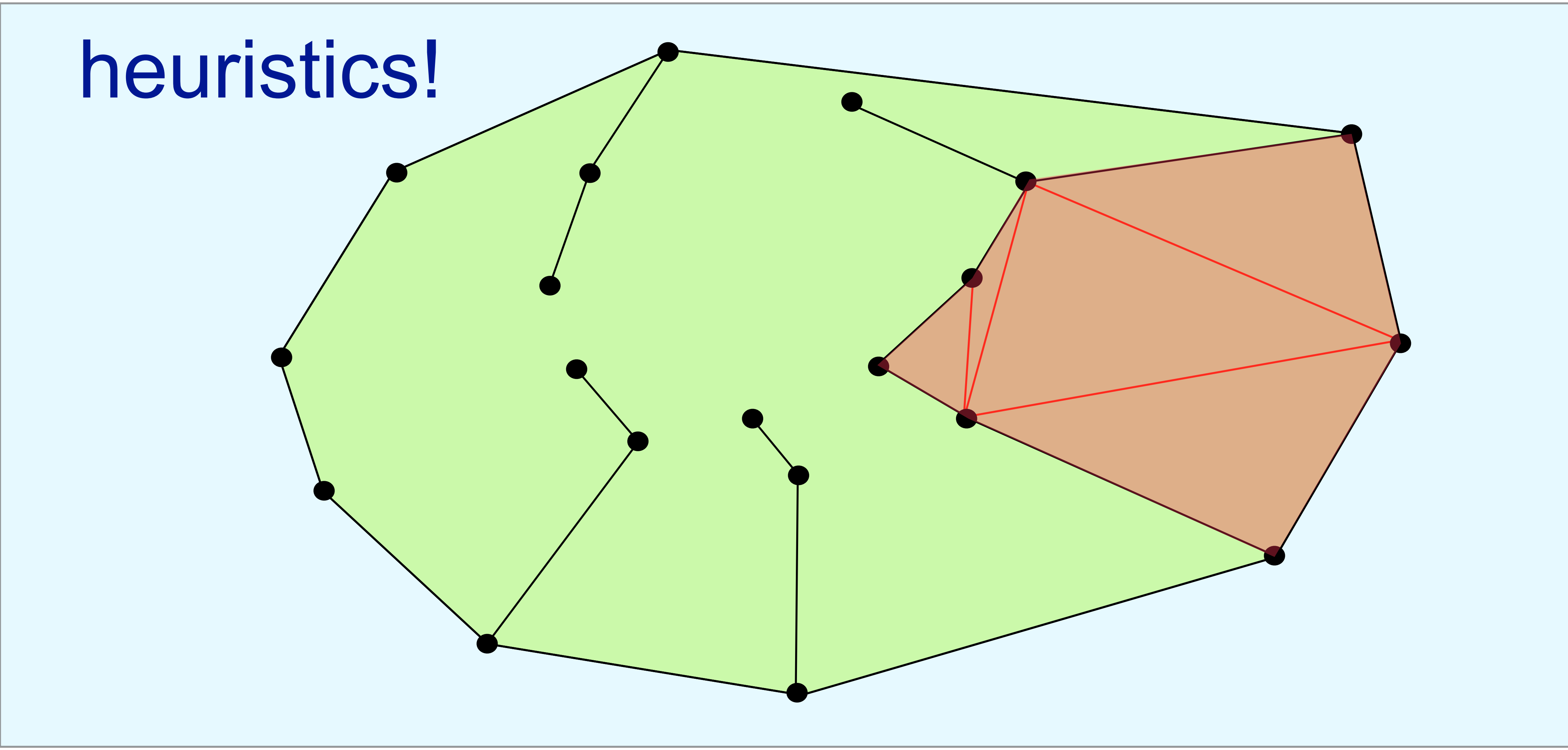


2. Use the heuristics to find more edges that have to be in the MWT.

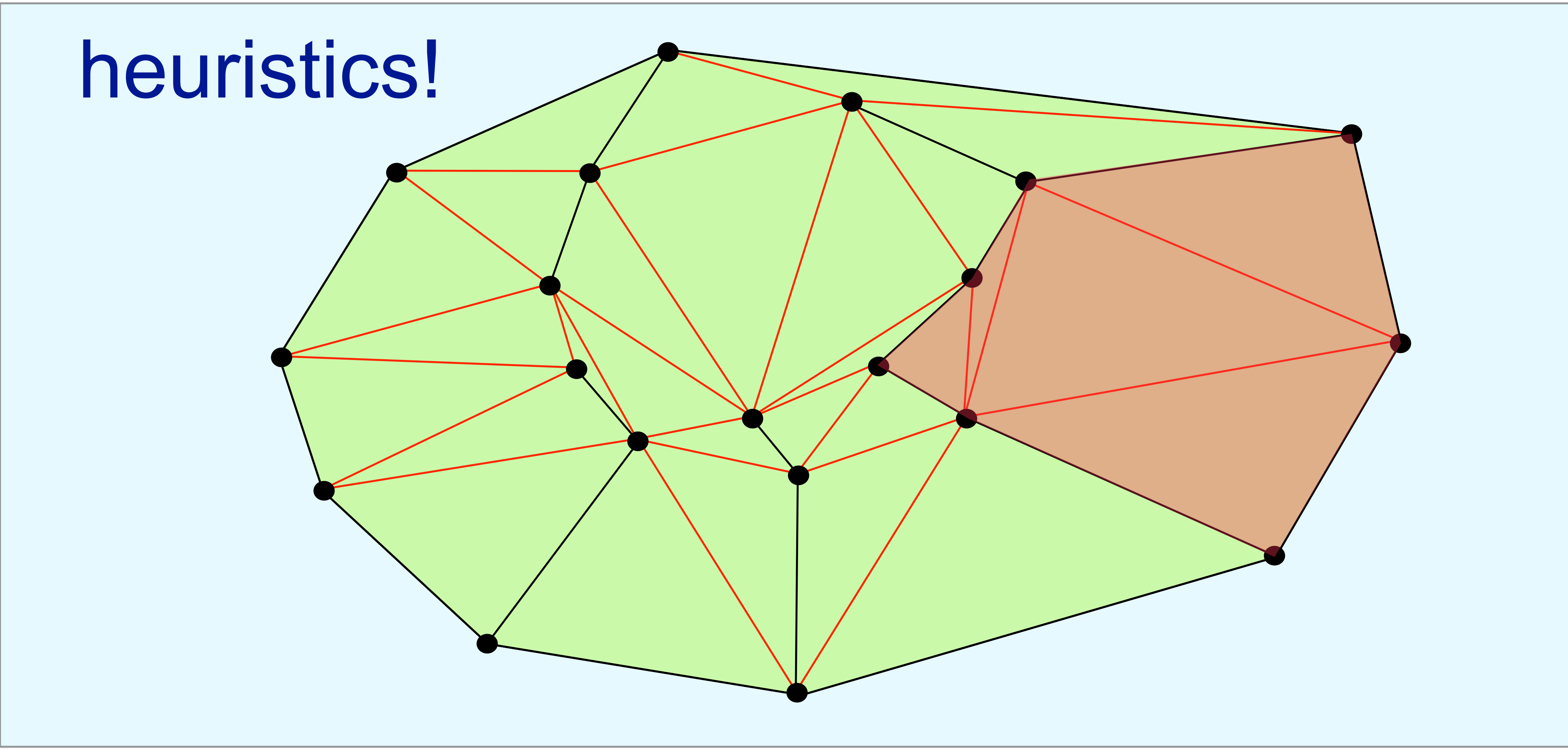
heuristics!



if you're lucky...
found edges connect all points to boundary.
Then remaining regions are simple polygons.



3. Triangulate each remaining region optimally using the dynamic-programming algorithm.



3. Triangulate each remaining region optimally
using the dynamic-programming algorithm.

heuristics

- This approach solves most random 40,000-point instances. [Dickerson et al. '97]

- But.. for random instances, heuristics leave (in expectation) $\Omega(n)$ internal components

  (but hidden constant is astronomically small, 10^{-51}).

  [Bose et al. '02]

linear programs for MWT

- [1985] Dantzig et al.
  Triangulations (tilings) and certain block triangular matrices.
- Subsequently studied in [1996 Loera et al; 2004 Kirsanov, etc...]

edge-based linear programs:

- [1997] Kyoda et al.
  A branch-and-cut approach for minimum weight triangulation.
- [1996] Kyoda.
  A study of generating minimum weight triangulation within practical time.
- [1996] Ono et al. *A package for triangulations.*
- [1998] Tajima. *Optimality and integer programming formulations of triangulations in general dimension.*
- [2000] Aurenhammer and Xu. *Optimal triangulations.*

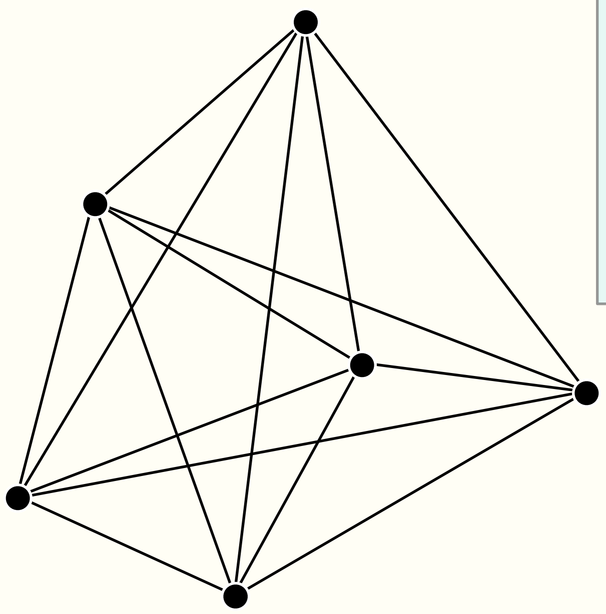
Dantzig et al's triangle-based LP [1985]

minimize $\sum_{\text{triangles } t} |t| X_t$

subject to

$$\sum_{t \ni p} X_t = 1 \quad (\forall \text{ points } p)$$

$$X_t \geq 0 \quad (\forall \text{ triangles } t)$$



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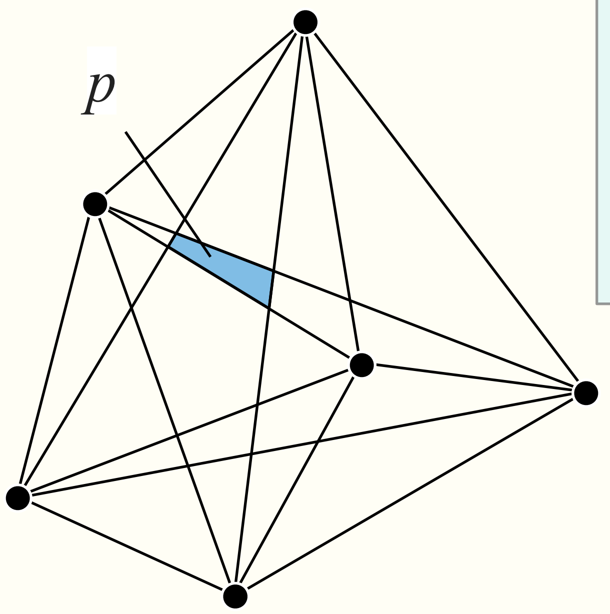

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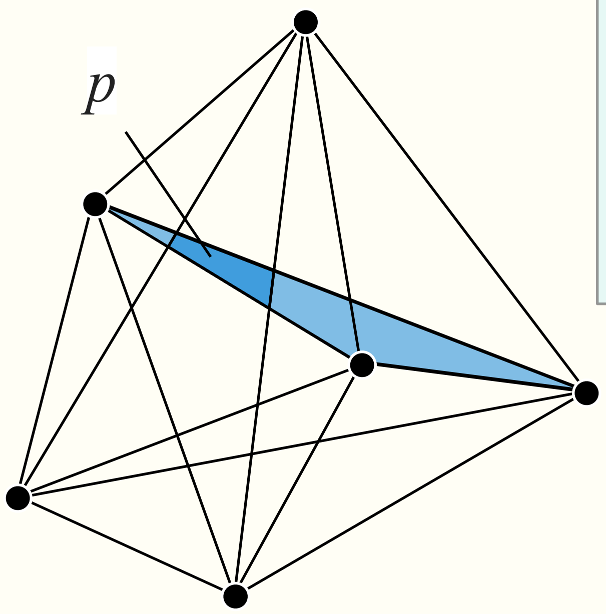

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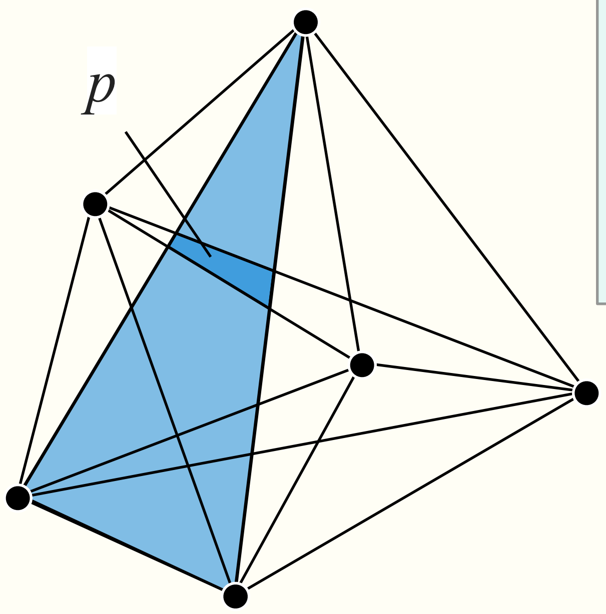

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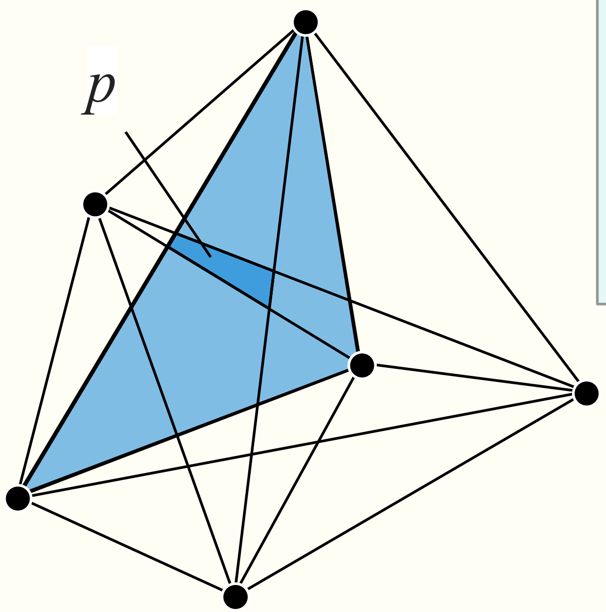

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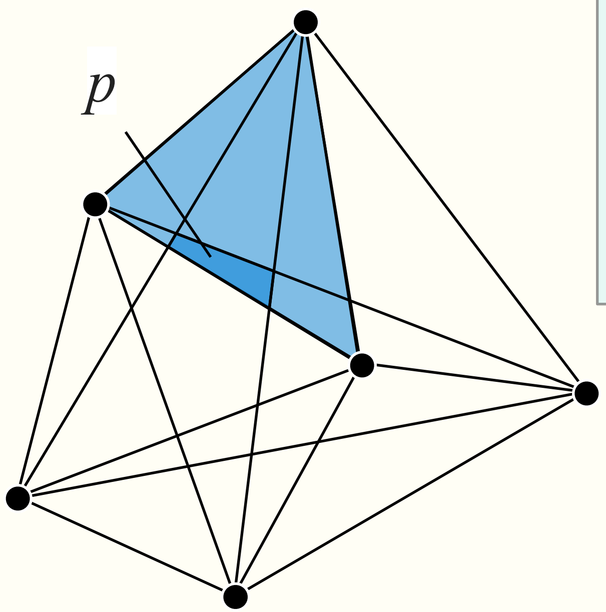

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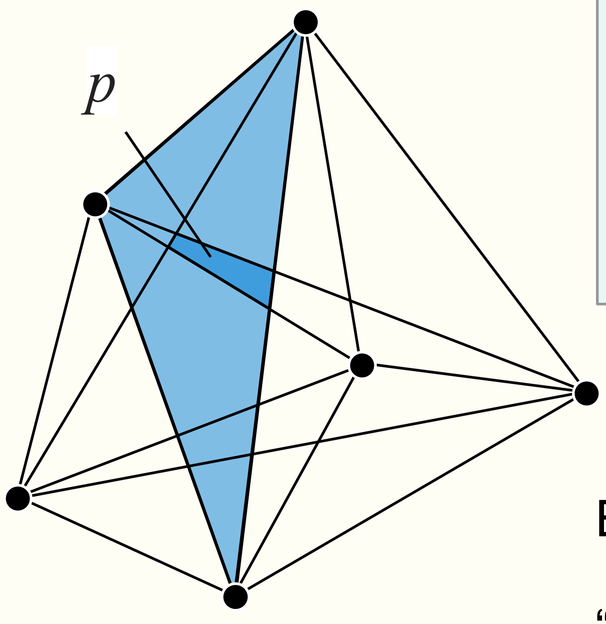

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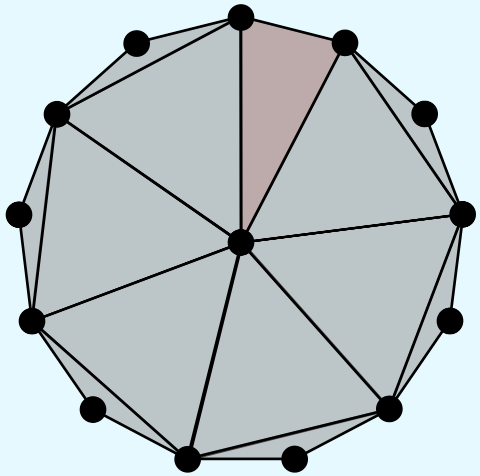
Exactly one of these triangles must be in the triangulation.

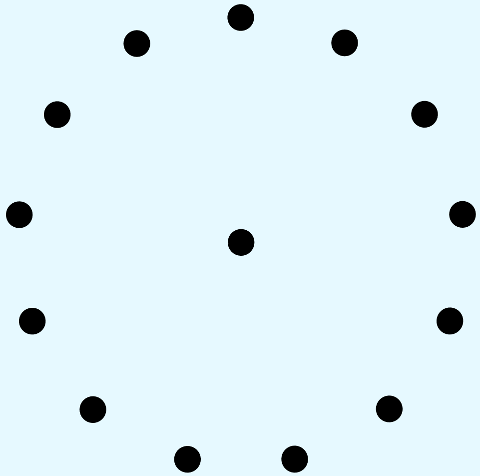
"Exact cover by triangles."

Integer vs. fractional MWT

Integer MWT



Fractional MWT
Each triangle has weight 1/2



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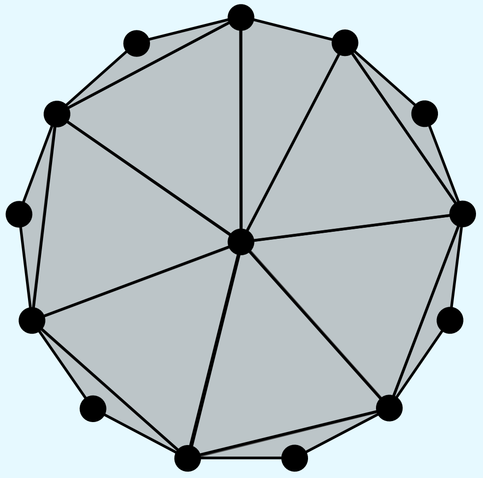

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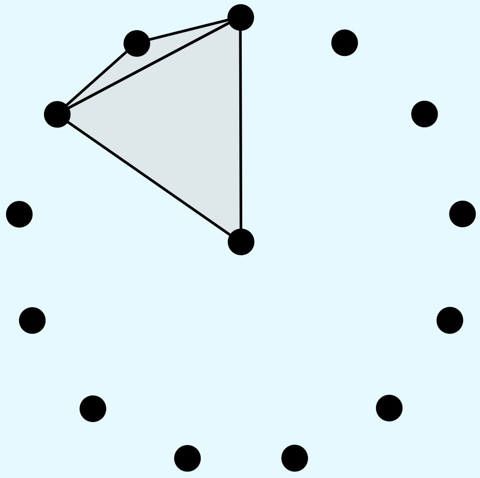

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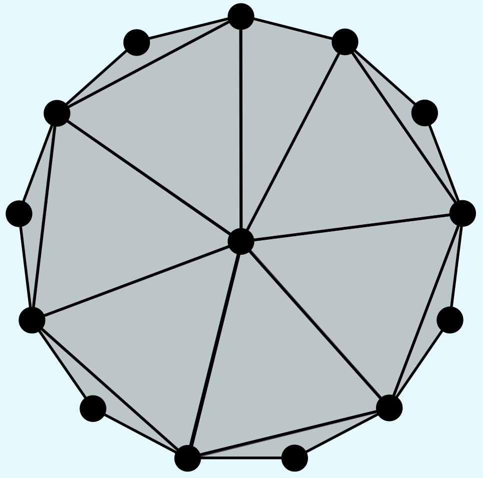

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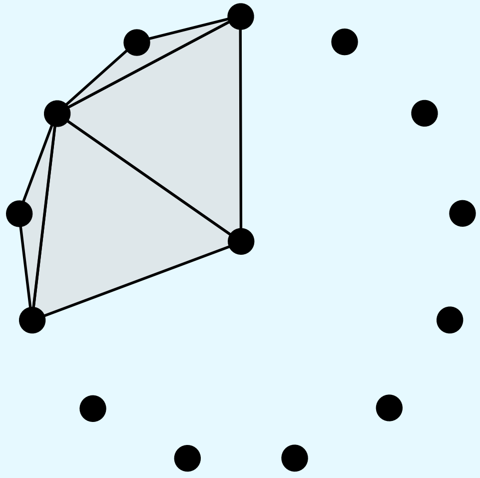

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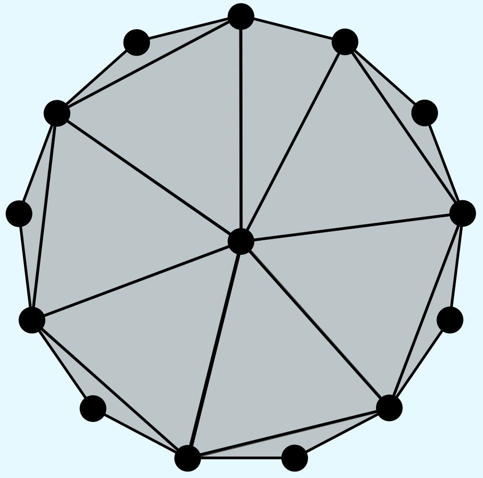

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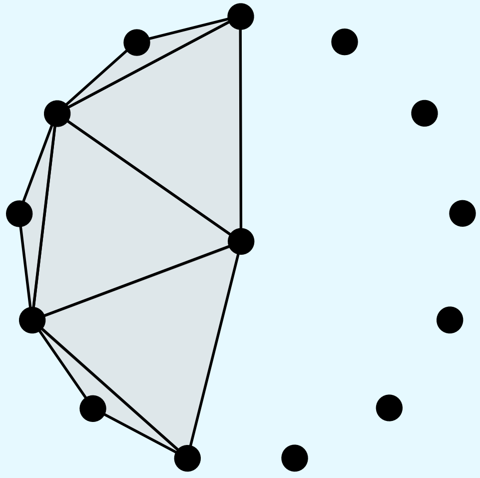

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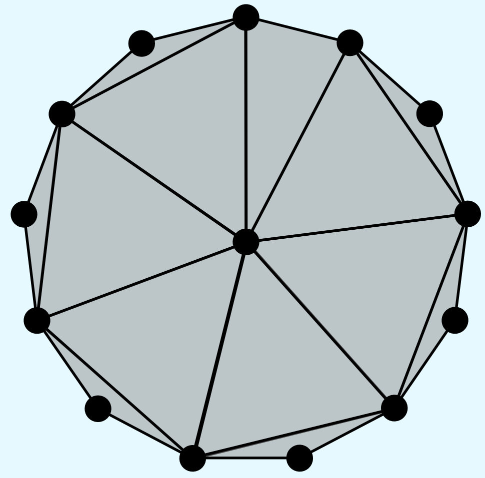

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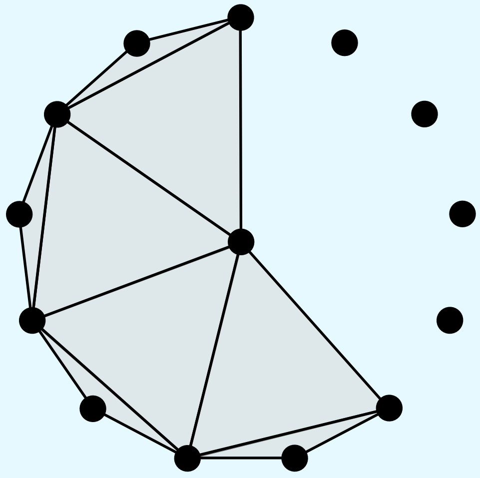

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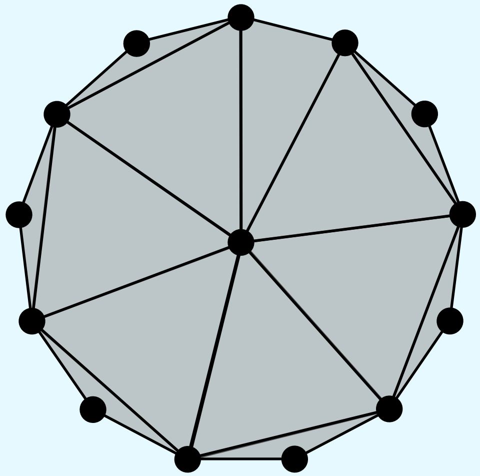

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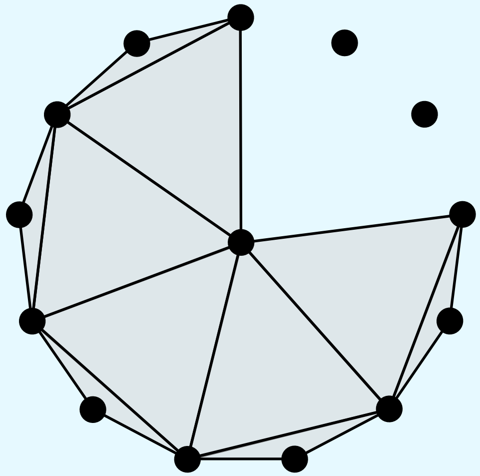

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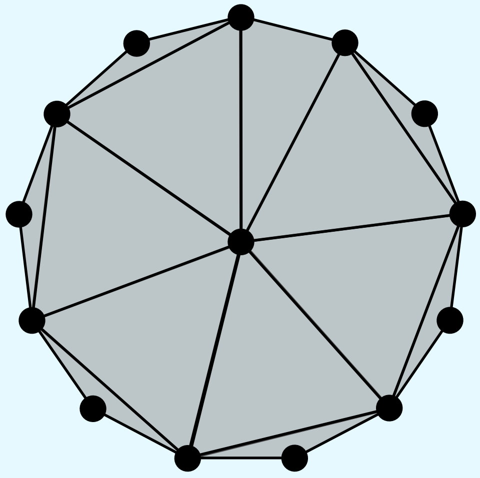

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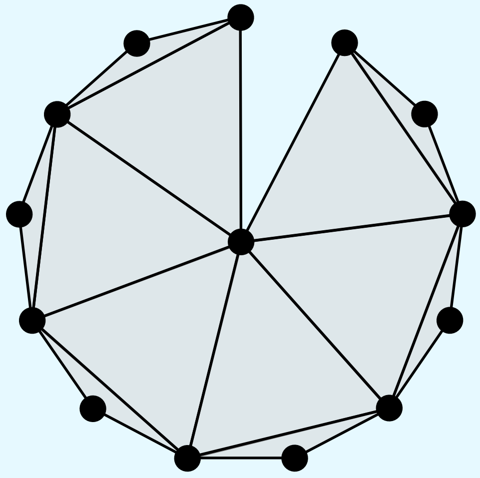

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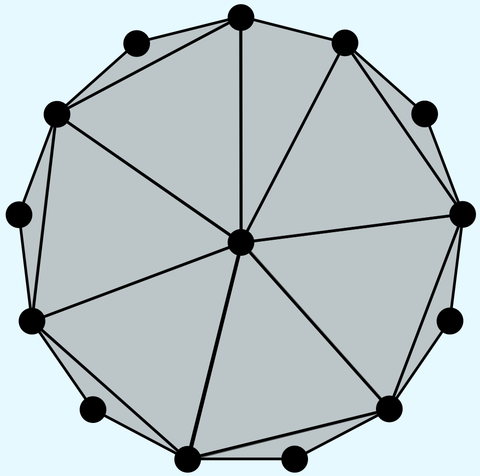

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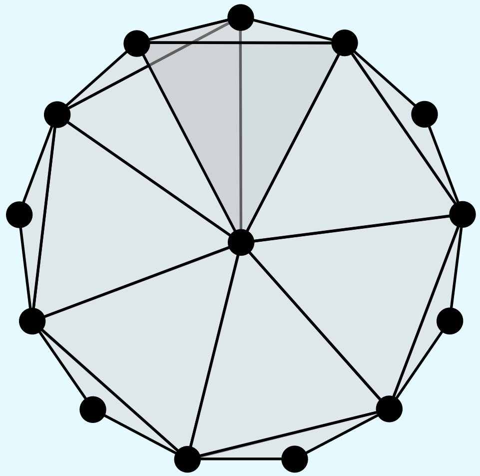

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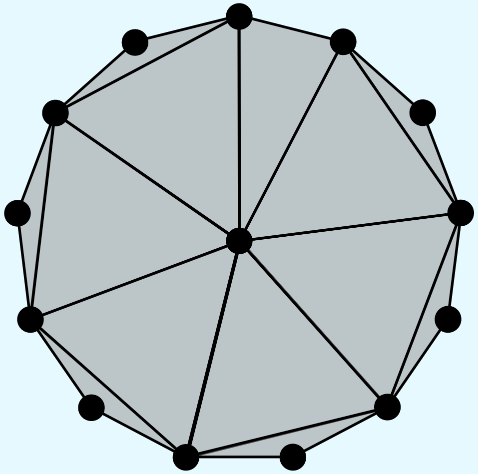

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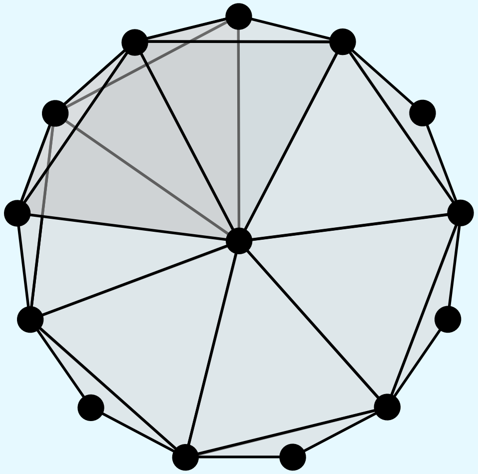

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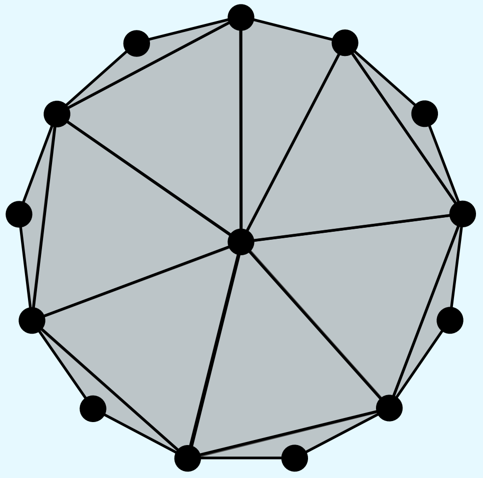

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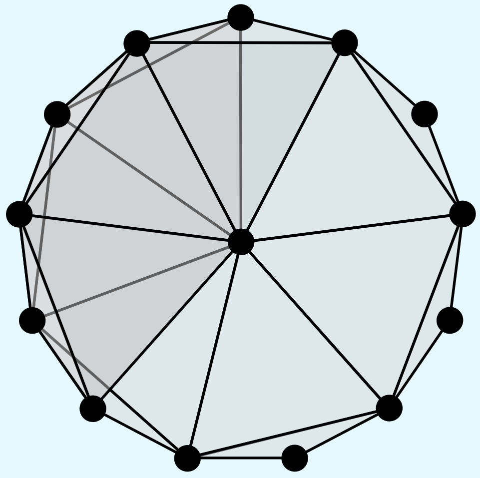

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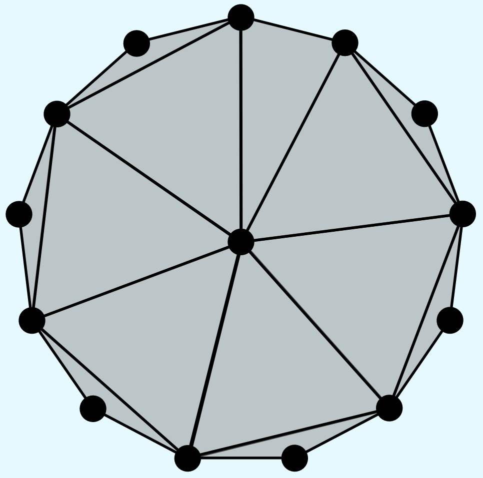

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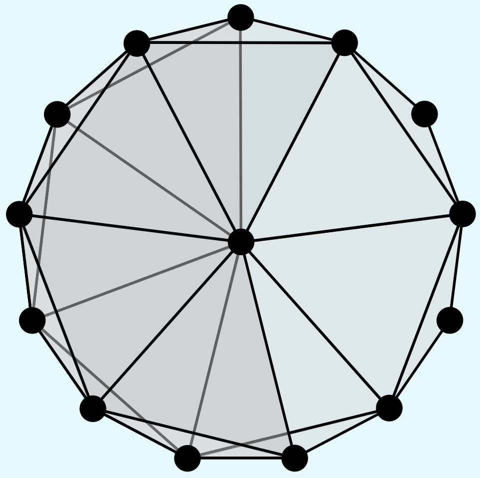

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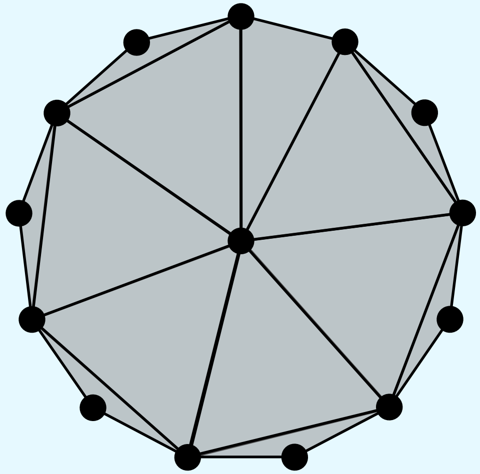

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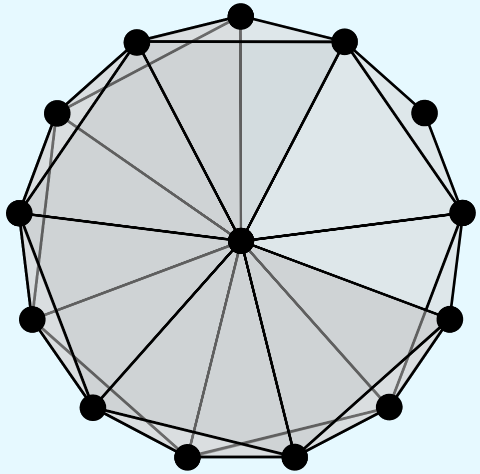

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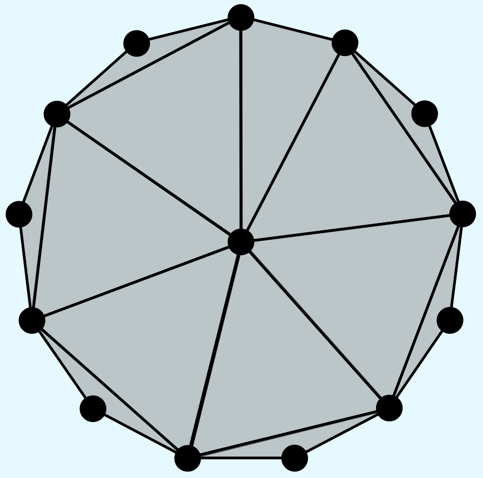

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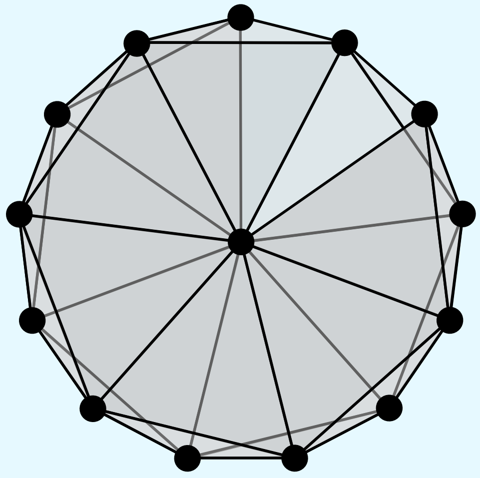

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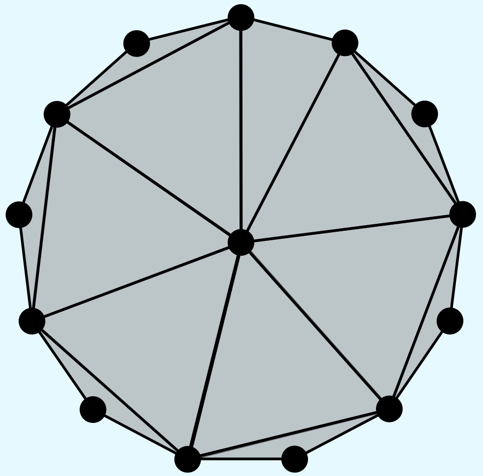

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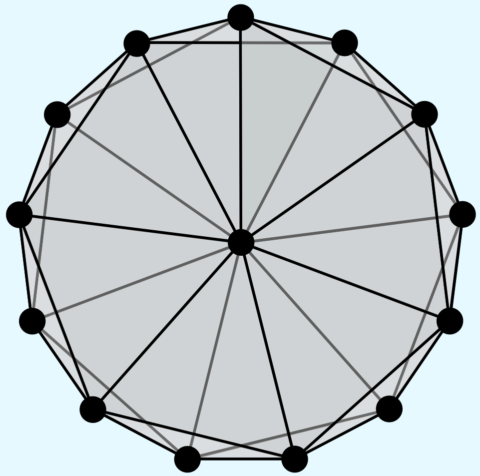

Ratio of costs is about 1.001

known results

- The integrality gap is at least 1.001 [2004 Kirsanov]
- For **simple-polygon** instances, the LP finds the MWT.
  [1985 Dantzig et al; 1996 Loera et al; 2004 Kirsanov; etc]

first new result

THM 1: *The integrality gap of the LP is constant.*

first new result

THM 1: *The integrality gap of the LP is constant.*

proof idea:

As Levcopoulos and Krznaric [1996] show, their algorithm produces triangulation T of cost at most

O(1) times the MWT (optimal integer solution).

We show that their triangulation T has cost at most

O(1) times the optimal ***fractional*** LP solution.

second new result

THM 2: *If the heuristics find the MWT for a given instance, then so does the LP.*

second new result

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proof idea:

If a heuristic shows that an edge is not in any MWT, we show that the optimal *fractional* triangulation cannot use the edge either.

If a heuristic shows that an edge is in every MWT, we show that the optimal *fractional* triangulation must use the edge fully as well.

Requires painstakingly adapting each analysis.

example

- Most heuristics based on local-improvement arguments.

For example, a heuristic might show (x,y) is in every MWT by contradiction. Suppose (x,y) is not in a given triangulation.

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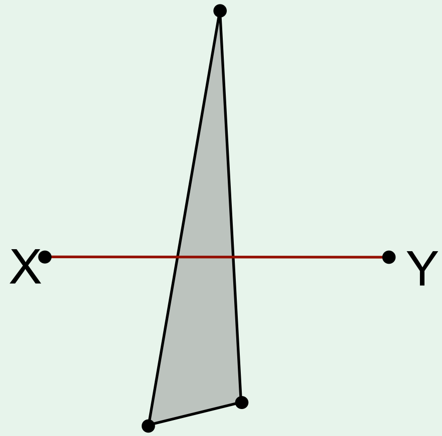
For example, a heuristic might show (x,y) is in every MWT by contradiction. Suppose (x,y) is not in a given triangulation.

Then some triangle in the triangulation must cross (x,y):



example

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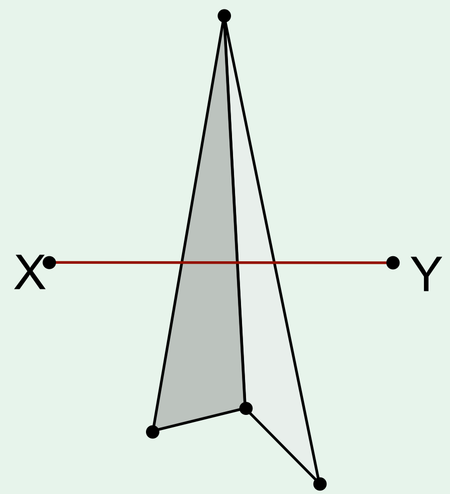
For example, a heuristic might show (x,y) is in every MWT by contradiction. Suppose (x,y) is not in a given triangulation.

Then some triangle in the triangulation must cross (x,y).

The triangulation must extend this triangle on each side:



example

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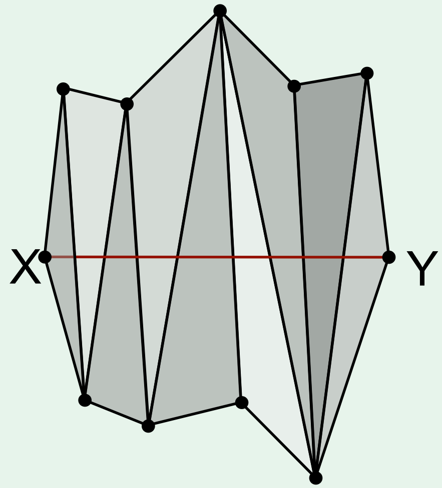
Then some triangle in the triangulation must cross (x,y).

Continuing, the triangulation covers (x,y) **locally** something like this:



example

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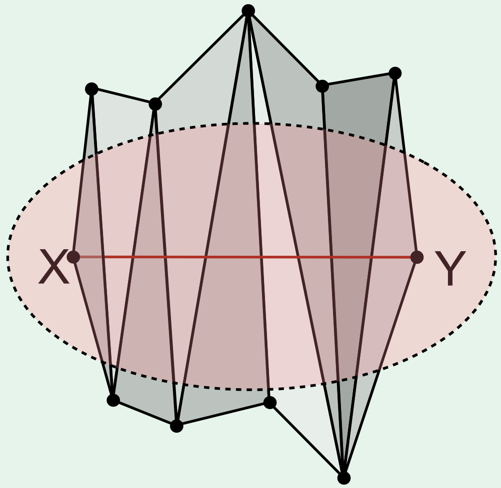
One shows that, given the heuristic condition, this triangulation can be improved, contradicting MWT.



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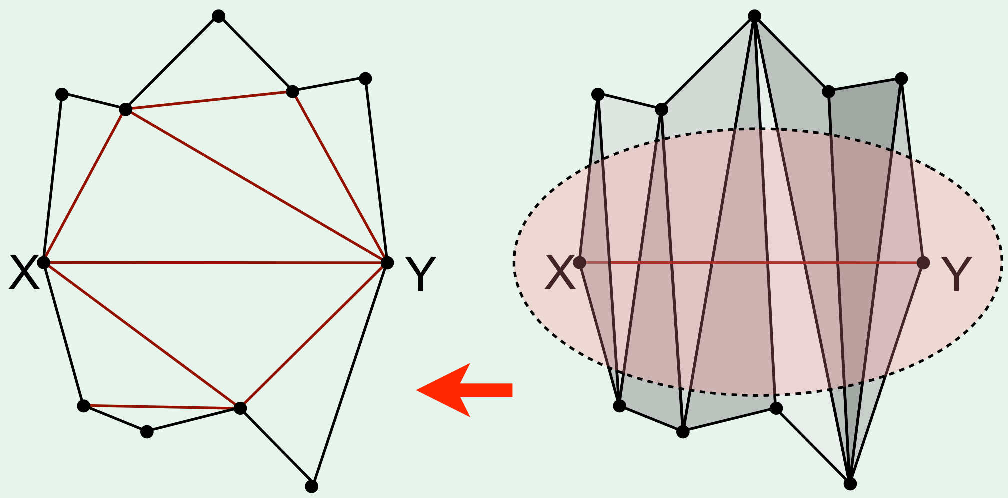
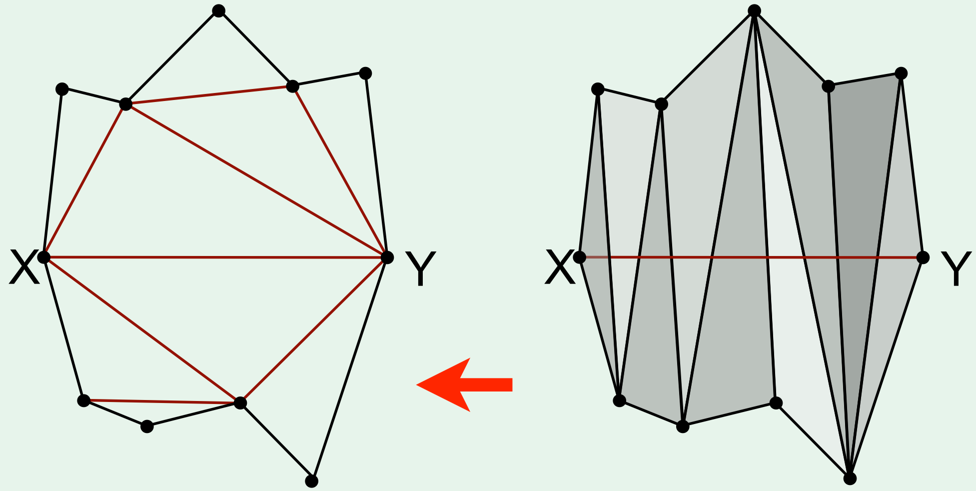
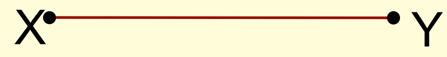


example - extending to fractional MWT

Assume for contradiction that (x,y) edge is not used fully (with total weight 1) in the fractional MWT.
Some triangle that crosses (x,y) must have positive weight.



extending to fractional triangulation



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Assume for contradiction that (x,y) edge is not used fully (with total weight 1) in the fractional MWT.
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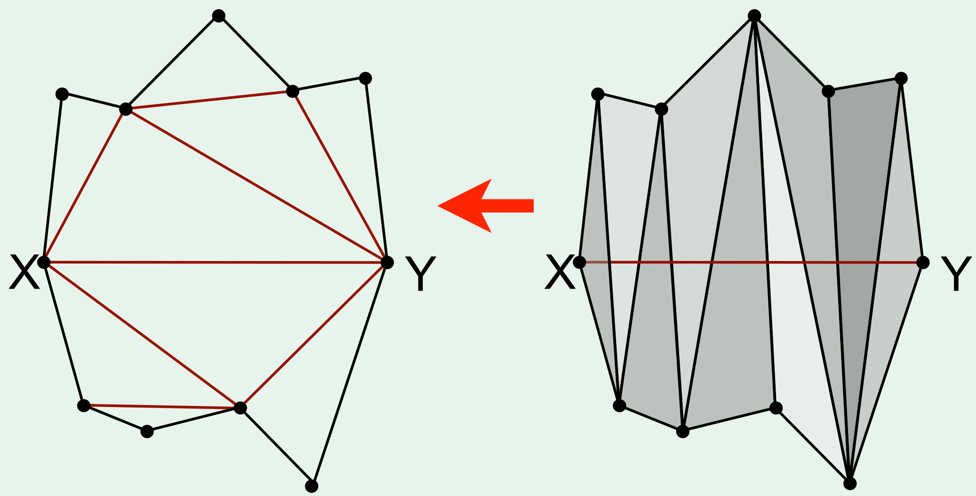
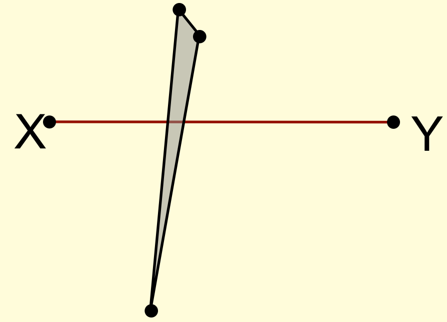
Can again find a sub-triangulation over (x,y) with positive wt.



extending to fractional triangulation

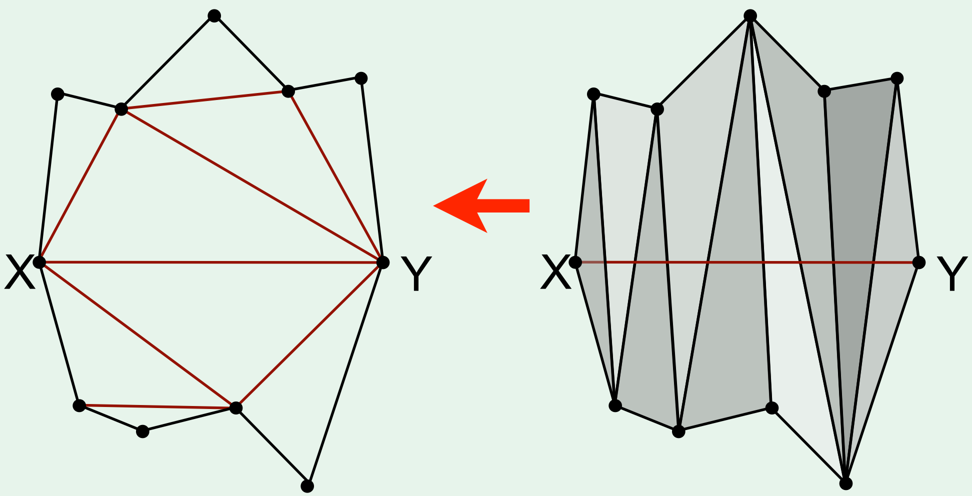


example - extending to fractional MWT

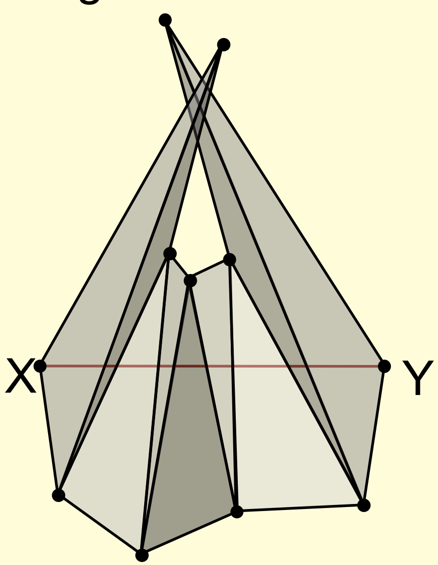
Assume for contradiction that (x,y) edge is not used fully (with total weight 1) in the fractional MWT.
Some triangle that crosses (x,y) must have positive weight.

Can again find a sub-triangulation over (x,y) with positive wt.
But the triangles covering (x,y) may overlap!

Complicates argument, but is not fatal.



extending to fractional triangulation



open problems

- What is the integrality gap of the LP? All we know:

$$1.001 \leq \text{integrality gap} \leq 54(\lambda + 1)$$

(λ is a very large constant.)

- Find an algorithm with **small** constant approximation ratio.

- Primal dual? Randomized rounding?

- Is there a PTAS?

  Do constantly many rounds of lift-and-project bring the integrality gap of the LP to $1+\varepsilon$?