A Lower Bound on the Size of Universal Sets for Planar Graphs

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Abstract

A *Fáry embedding* of a planar graph G is an embedding of G into the plane, no edges crossing, with each edge embedded as a straight line segment. A set $A \subseteq \mathbb{R}^2$ is said to be *n-universal* if every n-node planar graph has a Fáry embedding into A. We show that any n-universal set has size at least $1.098n$, for sufficiently large n.

1 Introduction

By a *Fáry embedding* of a planar graph G we mean an embedding of G in the plane in such a way that edges are mapped into line segments which can intersect only at endpoints. Fáry [F] proved that each planar graph has such an embedding. Other proofs of this fact, found independently, are given in [St, Wa].

For an integer n, a set A of points in the plane is called *universal for n-node planar graphs* (or just *n-universal* or *universal*) if every n-vertex planar graph G has a Fáry embedding into A, in that the vertices of G are mapped into points of A. By Fáry's theorem, the whole plane is universal, as is the unit square. In fact, by slightly modifying Fáry's proof, one can show that there exist *finite* universal sets. The sets obtained from this construction are of exponential size.

Until recently, whether it was possible to construct smaller universal sets, in particular universal sets of size n for n-node graphs, was an open problem. Rosenstiehl and Tarjan [RT] asked whether a polynomial-size grid could be universal. These problems were solved in a recent paper by de Fraysseix, Pach and Pollack [FPP]. They proved that the $(2n-4)\times(n-2)$ grid is universal, thus showing that universal sets may have polynomial size. They also

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proved the first non-trivial lower bound: universal sets cannot have $n + o(\sqrt{n})$ points; $o(\sqrt{n})$ additional points are insufficient. An independent construction of an embedding into the $(2n-4) \times (2n-4)$ grid was given by Schnyder [Sc], who also announced recently that he can decrease the size of the grid to $(n-1) \times (n-1)$. It is not hard to see that a universal grid must have size at least $n/3 \times n/3$.

We improve the lower bound from [FPP] by proving that each universal set must have size at least $1.098n$; *linearly* many extra points are indeed necessary.

2 The Lower Bound

We first construct an auxiliary graph G and show that G has few Fáry embeddings. Let k be an integer whose value will be specified later. Let G be a k-vertex graph constructed as follows. The vertices of G are $\{v_1, v_2, \ldots, v_k\}$. G contains a $k-2$-cycle $(v_1, \ldots, v_{k-2})$ and all v_i in this cycle are joined to v_{k-1} and v_k. There are no other edges in G.

Lemma 1 *Let A be a finite set of points in the plane and let $a = |A|$. Then G has at most $6(2k-4)(k-3)\binom{a}{k}$ Fáry embeddings into A.*

Proof. Choosing an embedding of G can be divided into two phases: choosing the set $B \subseteq A$ of k points onto which G is mapped, and mapping the vertices of G onto the points in B. Clearly the choice of B can be made in at most $\binom{a}{k}$ ways. So now we concentrate on embedding G onto B.

Note that since all faces of G are symmetric, it does not matter which face is chosen as the outer face; all embeddings are homeomorphic. The outer face can be chosen in $2k-4$ ways. Suppose that we chose face $\{v_1, v_{k-1}, v_{k-2}\}$ to be the outer face. Clearly the convex hull of B must be a triangle. The vertices v_1, v_{k-1}, v_{k-2} can be mapped onto the triangle in at most 6 ways. Then we have at most $k-3$ choices for v_k. Having already fixed the embedding of v_1, v_{k-1}, v_{k-2} and v_k, the other vertices have at most one possible embedding. ■

Now we construct a family $\mathcal{G}$ of labeled graphs on vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $U = \{v_1, v_2, \ldots, v_k\}$ and treat G as a graph with vertex set U. Number G's faces $1, 2, ..., 2k-4$ in some arbitrary order. For each vector $\bar{x} = (x_1, x_2, \ldots, x_{2k-4})$ of non-negative integers satisfying $\sum_{i=1}^{2k-4} x_i = n-k$, $\mathcal{G}$ contains one triangulated graph $G(\bar{x})$ which has the properties that the restriction of $G(\bar{x})$ to U is G, and that there are exactly x_i vertices in face number i of G. $\mathcal{G}$ contains no other graphs than these. It is easy to see that

$$|\mathcal{G}| = \binom{n+k-5}{2k-5}.$$

Suppose now that a set A with $|A| = a$ is n-universal. We will show that if a is too small, the graphs from $\mathcal{G}$ cannot all be embedded in A. Fix an embedding $f_{\bar{x}}$ for each $G(\bar{x})$. By Lemma 1, we know that there are at least

$$\frac{\binom{n+k-5}{2k-5}}{6(2k-4)(k-3)\binom{a}{k}} \tag{1}$$

functions $f_{\bar{x}}$ which are equal on U.

Lemma 2 *Each embedding f of G can be extended to at most*

$$\binom{a-n+2k-5}{2k-5} \tag{2}$$

functions $f_{\bar{x}}$.

Proof. For a given extension $f_{\bar{x}}$ of f, let r_i be the number of points in the image of face number i which were *not* chosen by $f_{\bar{x}}$. Then we have $r_1 + r_2 + \cdots + r_{2k-4} = a - n$. A different extension $f_{\bar{x}'}$ would give rise to a different sequence $r'_1, r'_2, ..., r'_{2k-4}$. So the number of possible extensions is at most the number of solutions of this equation with all $r_i \geq 0$. This immediately yields the lemma. ■

From (1) and (2),

$$\binom{a-n+2k-5}{2k-5} \geq \frac{\binom{n+k-5}{2k-5}}{12k^2\binom{a}{k}}. \tag{3}$$

Let us now take $a = \lfloor (1+\epsilon)n \rfloor$ and $k = \lfloor \delta n \rfloor$ where ϵ and δ are constants. Inequality (3) implies

$$\binom{\lfloor \epsilon n \rfloor + 2\lfloor \delta n \rfloor - 5}{2\lfloor \delta n \rfloor - 5} \geq \frac{\binom{\lfloor (1+\delta)n \rfloor - 5}{2\lfloor \delta n \rfloor - 5}}{12\lfloor \delta n \rfloor^2 \binom{\lfloor (1+\epsilon)n \rfloor}{\lfloor \delta n \rfloor}}. \tag{4}$$

From Stirling's formula as applied to (4), we can infer that there is a polynomial $p(n)$ satisfying

$$\left(\frac{(\epsilon+2\delta)^{\epsilon+2\delta}}{\epsilon^{\epsilon}(2\delta)^{2\delta}}\right)^n \geq \frac{1}{p(n)} \left(\frac{(1+\delta)^{1+\delta}}{(2\delta)^{2\delta}(1-\delta)^{1-\delta}} \frac{\delta^{\delta}(1+\epsilon-\delta)^{1+\epsilon-\delta}}{(1+\epsilon)^{1+\epsilon}}\right)^n$$

for sufficiently large n. But for $\epsilon = 0.098$ and $\delta = 0.133$, one can show that in fact

$$\frac{(\epsilon+2\delta)^{\epsilon+2\delta}}{\epsilon^{\epsilon}} < \frac{(1+\delta)^{1+\delta}}{(1-\delta)^{1-\delta}} \frac{\delta^{\delta}(1+\epsilon-\delta)^{1+\epsilon-\delta}}{(1+\epsilon)^{1+\epsilon}},$$

so we have a contradiction for large enough n. Therefore we obtain

Theorem 1 *If A is n-universal, then for n large enough, $|A| > 1.098n$.*

3 Remarks

There is still a very large gap between the upper and lower bound for universal sets. Finding an asymptotically tight estimate for the minimum size of universal sets is an interesting open problem. This problem appears, however, to be quite difficult. It would be nice to have a superlinear lower bound or a subquadratic upper bound.

The methods used in this note cannot give superlinear lower bounds. In fact, a careful analysis of this technique shows that it cannot give a lower bound better than $2n$. This bound is obtained by assuming that G has, miraculously, only *one* embedding. Because it *is* possible to construct a set A in which G has, roughly, $\binom{a}{k}$ embeddings, the bound from Lemma 1 is close to optimal.

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