

Floors and Ceilings	
• For any real number x,	
	r less than or equal to x eater than or equal to x
• For any integer n,	
n/2 + n/2 = n	
• For integers a 0 and b 0,	
n/a /b = n/(al	b)
n/a /b = n/(al	
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# Logarithms

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Definition: For any a, b , c:
•
             log<sub>b</sub> a = c
                                   b<sup>c</sup> = a
    We use:
•
             lg n = log<sub>2</sub> a
                                   (binary logarithm)
             ln n = log<sub>e</sub> a
                                   (natural logarithm)
   Properties (writing log for a logarithm with arbitrary base):
•
                       = b<sup>logb a</sup>
             ۵
             log (a b)= log a + log b
             log a<sup>n</sup> = n log a
             \log_b a = (\log_c a) / (\log_c b) (*)
             log (1/a)= - log a
             log_b a = 1/log_a ba^{log_b n} = n^{log_b a}
• (*) implies that e.g. (\lg n) = (\log_c n) for any c.
    The base of the logarithm is irrelevant for asymptotic analysis!
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Forward Substitution Method		
Ü Guess a solution.		
Ü Verify by induction.		
<ul> <li>For example, for</li> </ul>		
T(n) = 2 T( n / 2 ) + n and	T(1) = 1	
we guess T(n) = O(n lg n)		
Induction Goal:		
T(n) cnlgn, for some c	and all n > n <sub>o</sub>	
Induction Hypothesis:	·	
T(n/2) cn/2 lg n/	/ 2	
<ul> <li>Proof of Induction Goal:</li> </ul>		
T(n)=2T(n/2)+n 2(cn/2 lgn/2 cnlg(n/2)+n =cnlgn-cnlg2+n =cnlgn-cn+n		
c n lg n	provided c 1	47

# ... Forward Substitution Method

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• So far the restrictions on c, n_0 are only c 1

• Base Case:

T(n_0) c n lg n

Here, n_0 = 1 does not work, since T(1) = 1 but c 1 lg 1 = 0.

However, taking n_0 = 2 we have:

T(2) = 4 2 lg 2 = 2

so

T(2) c 2

holds provided c 2.
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Linearity: ( k   1 k n·ca <sub>k</sub> +b <sub>k</sub> ) = c ( k   1 k n·a <sub>k</sub> )+( k   1 k n·b <sub>k</sub> )	
= c( k   1 k $n \cdot a_k$ ) + ( k   1 k $n \cdot b_k$ )	
Lize for any unstatic notations	
Use for asymptotic notation:	
$(k 1 k n \cdot (f(k))) = (k 1 k n \cdot f(k))$	
In this equation, the -notation on the left hand side applies to variable k, but on the right-hand side, it applies to n.	
Arithmetic Series:	
$(k   1 k n \cdot k) = n(n+1) / 2$ = $(n^2)$	
Geometric (or Exponential) Series: If $x = 1$ then	
$(k \mid 0 k n \cdot x^{k}) = (x^{n+1} - 1) / (x - 1)$	
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#### ... Summations

- Infinite Decreasing Geometric Series: If |x| < 1 then (  $k \mid 0 \quad k < \quad \cdot x^k$ ) = 1/(1-x)
- Harmonic Series:

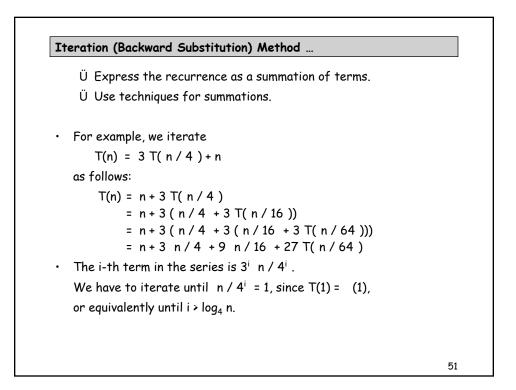
 $H_n = 1 + 1/2 + 1/3 + ... + 1/n$ = (k|1 k n · 1/k) = ln n + O(1)

 Further series obtained by integrating or differentiating the formulas above.

For example, by differentiating the infinite decreasing geometric series and multiplying with x we get:

$$(k \mid 0 k \cdot kx^{k}) = x/(1-x)^{2}$$

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### ... Iteration (Backward Substitution) Method

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• We continue:

T(n) = n+3 n/4 + 9 n/16 + 27 T(n/64)
n+3 n/4 + 9 n/16 + 27 n/64 + ... + 3^{\log_4 n} (1)
\{as a^{\log_b n} = n^{\log_b a}\}
n(i|0 i < \cdot (3/4)^i) + (n^{\log_4 3})
\{decreasing geometric series:
(k|0 k < \cdot x^k) = 1/(1-x)\}
4 n + (n^{\log_4 3})
\{\log_4 3 < 1\}
= 4 n + o(n)
= O(n)
```

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### Merge Sort with the Master Theorem

